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(Contents Continued on Page 2509)

(Contents Continued from Front Cover)

TECHNICAL CORRESPONDENCE

Subnormal Distribution Derived From Evolving Networks With Variable Elements

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***Abstract*—During the past decades, power-law distributions have played a significant role in analyzing the topology of scale-free networks. However, in the observation of degree distributions in practical networks and other nonuniform distributions such as the wealth distribution, we discover that, there exists a peak at the beginning of most real distributions, which cannot be accurately described by a monotonic decreasing power-law distribution. To better describe the real distributions, in this paper, we propose a subnormal distribution derived from evolving networks with variable elements and study its statistical properties for the first time. By utilizing this distribution, we can precisely describe those distributions commonly existing in the real world, e.g., distributions of degree in social networks and personal wealth. Additionally, we fit connectivity in evolving networks and the data observed in the real world by the proposed subnormal distribution, resulting in a better performance of fitness.**

***Index Terms*—Degree distribution, evolving networks, Gibrat's law, power-law distribution, probability theory.**

I. Introduction

As well known, the power-law distribution is a nonuniform distribution which appears that a majority of vertices hold a low number of links while a few vertices have many links for the networks. The history of the power-law distribution starts from the Italian economist Pareto in the 19th century, who first put the "20–80" rule forward, i.e., 20% of a population possess 80% social welfare, apparently following a power-law distribution. Bababási first employed the power-law distribution to explain the degree distribution of scale-free (SF) networks and made it gain considerable fame. In 1999, he revolutionarily evolved the network model into a scale-invariant state with the growing and preferential attachment character, and revealed that the degree distribution of evolving networks follows a power-law distribution [1].

This discovery soon drew great attentions from many multidisciplinary researchers and brought a stirring of interests in evolving networks. Many models based on evolving networks have been proposed and well studied based on Bababási's work, such as fitness models [2], the nonlinear accelerating networks [3], copying networks [4], weight-driven local-world networks [5], optimal networks [6], etc. Apart from these models, it is well known that most practical networks such as Web networks [7], interaction networks [8], sorting comparison network [9], social networks [10], [11], etc., all follow a power-law distribution, which can be described as the "rich-get-richer" or Matthew effect. The power-law degree distribution therefore has shown significance in the study of complex systems and is the foundation of exploring the formation mechanism and organizational principle of SF networks. Inspired by the SF networks along with their properties such as its power-law distribution, the discovery, analysis, and application of complex systems are rapidly stepping into a new stage. Some of those outstanding achievements from various fields in recent years are listed below: analyzing the significance of a social community by integrating the specific characteristics of social networks [12], mining the social community structure by integrating center locating and membership optimization [13], seeking the solution of social dilemmas on evolving random networks [14], reviewing the co-evolutionary games [15] and computation, proposing control protocols to estimate finite settling time for complex systems [16], achieving the application of fast search between any two vertices by utilizing the nonuniformity property of SF network degree distribution [17] and studying the decentralized adaptive pinning-control for cluster synchronization of complex networks using a local adaptive strategy [18], and solving the robust synchronization problem of fuzzy complex dynamical networks [19], all of these work make the complex network researches a "new science of networks." Specifically, among all the complex network researches, many scientists have devoted to the study of degree distribution of evolving networks. They proposed mean-field [20], master-equation [21], and Markov-chain approach [22] to mathematically solve the degree distribution. Recently, the deviations from power-law distributions were deeply investigated due to the facts of proportional growth and preferential attachment regarding practical networks [23].

In the past years, the power-law distribution has a dominated position in the field of network science. However, some

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researchers recently started doubting whether it fits in all kinds of networks. As one of them described in [24]: both real networks as the connectivity server 2 and the *m*-varying model are obviously different from the power-law distribution. And we precisely notice that most practical SF networks and other dynamical systems are in accord with the rule rich-get-richer, however, many of them are out of accord with "poor-get-poorer," indicating that the distributions of these systems are not simply power-law. This phenomenon appears widely in complex networks and dynamical systems as well. For aerial inbound logistical operation, there obviously exists a peak in each distribution, the shape of the left part closes to a Poisson distribution, and the right part has a heavy-tailed scaling statistics for the empirical data [25]. Another evidence [26] also indicates that many human activities including the interacting communication present a different type of interevent time distribution, neither completely Poisson nor power-law, but a bimodal combination of them. By investigating all these evidence and facts, we find out that the mode does not occur in the lowest value for SF networks and other dynamical systems. As a consequence, the plot presents a peak. Here, we name this phenomenon "subnormalization," as the curve of corresponding distribution seems to fall somewhere between the power-law and normal distribution.

In order to explain the origin of this phenomenon and find a better distribution function for the real world, we originally put forward a subnormal distribution, concretely introduce certain variable elements to the modeling process of SF network and employ some common mathematical methods to solve the distribution function of degree. Through the mathematical analysis, we find out this obtained subnormal distribution processes the properties of both power-law and log-normal distribution. By proposing this distribution, we aim to simulate the distributions of both evolving models and real networks. Therefore, in simulations, we compare the similarity between the network distribution and the proposed distribution. In addition, the distributions of degree in collaboration networks and personal wealth are also compared with our distribution to show that it can be fit in with the practical ones. Finally, we try to find out the mechanism of SF networks and discuss the potential value of subnormal phenomenon and subnormal distribution in other fields such as economics.

The organization of this paper is as follows. A detailed presentation of the evolving networks with variable elements is provided in Section II. The derivations of subnormal distribution and its statistical properties are presented in Section III. Simulations are carried out in Section IV to demonstrate the fitness to other distributions. Finally, some discussions, the conclusions, and outlooks are given in Section V.

II. Evolving Networks With Variable Elements and Its Degree Distribution

For the purpose of obtaining a better distribution than the power-law, we first carry out an evolving network model with variable elements based on an adapted SF network to study its degree distribution. As a common sense, SF networks suggest the growth and preferential attachment and follow a perfect power-law distribution. Though the Barabási and Albert (BA) network is closer to the real networks than other network models, the construction process is still too ideal. For real networks, it is impossible to introduce one vertex each time and connect it to m existing vertices. Contrarily, there exists lots of variable factors, such as the famous World Wide Web that has a variable vertex growth rate and edge connection. To reveal these influences on the degree distribution of networks, we discuss the variable elements and show the construction process of an SF model with them in this section.

A. Variable Elements

First, the variable elements in the process of construction of an SF network are discussed.

The initial network is one of the most negligent issue which in fact is also ignored by the BA SF network. The variable elements of the initial network are the number of initial vertices and their connection rules. As we know, the number of initial vertices affects the final degree distribution if the number is very huge. However, the initial network are always very small comparing to the final network. Advanced Research Projects Agency Network, the origination of Internet, has only four host computers connecting to each other. But now Internet has billions of computer connections. Therefore, the number of an initial network is required to be small enough that it does not affect the final structure of the network. In addition, the small initial networks are always highly gathered, e.g., the beginning of a new journal network is cited by each other and its average short path distance is low. Considering both small and low distance characteristics, we suppose that a small-world network proposed by Newman and Watts (NW) is appropriate to describe the initial network [27].

The other variable element is the arrival rate or interval time of vertices which is assumed as a constant by many theoretical models. BA networks, for example, suppose that one vertex is connected to the network each time. We suggest that the vertices introduced to the network should follow a certain rate, and that rate should vary in different period, i.e., it is related to time. Specifically, the growth rate during financial crises is lower than during boom period. Thus, a nonhomogeneous Poisson process can perfectly express the generation of vertices. In [28], we have proven that a nonhomogeneous Poisson process is irrelevant to the final degree distribution. Specifically, the size of an evolving network and its relative rate of growth are independent, which is precisely consistent with the famous Gibrat's law in economics [29]. In other words, this law is also valid for evolving networks.

The most significant variable in this paper is the connection of new arrival vertices which directly affects the degree distribution of the network. Accelerating growth network model [3] first notes that the connection is varied, and its connection follows a power-law distribution. However, for this kind of network, a stationary distribution of the degree distribution does not exist, which disagrees with the practical networks. The stationary distribution for connections of a network is

TABLE I
SUMMARY OF VARIABLES

Variable	Description
$N(t)$	The number of vertices at time t
m	The number of connections for each new vertex
$\mu\&\sigma$	Parameters for a log-normal distribution
$\lambda(t)$	Intensity function for a nonhomogeneous Poisson process
$s(t)$	Mean function for a nonhomogeneous Poisson process
$k_i(t)$	Degree of the vertex i at time t
$P_k(t)$	The degree distribution at time t

one of the goals of this paper. We mainly consider two kinds of distributions, uniform distributions and nonuniform distributions. The uniform distributions refer to those networks whose connections of new arrival vertices are homogeneous, hierarchical networks, e.g., [30]. On the contrary, the most common connection rule follows a nonuniform distribution, since each new vertex has its own fitness to the network, to which the number of its connections to the existing vertices in the networks relates. Practically speaking, in the network of the research reference, once a high quality paper such as *Emergence of scaling in random networks* by Barabási is published, many related references will emerge in a short time, yet a low quality paper will not be cited in a long time. The universal expression for this connection is the Gaussian distribution, most connection numbers are a mean value, and the extremely high and low numbers are rare which is much more common in a steady network than the power-law distribution. Furthermore, the Gaussian distribution is just an ideal situation. There are also many dilemmas by applying this distribution, e.g., the connection cannot be negative. There is evidence that the most income is distributed log-normally, which can be also interpreted as the new added connection, the income of evolving networks, is also distributed log-normally [31]. Taking all factors into consideration, we employ the log-normal distribution to simulate the connection of evolving networks in this paper. The log-normal distribution has a significant influence on the degree distribution of complex networks, which makes it free of time, but disobey the traditional power-law distribution and break the rule of poor-get-poorer.

There are other variable elements, such as the connection rule and time, which are not discussed in this paper.

B. Evolving Network Model Based on Variable Elements

Next, we show a constructing process of evolving networks with variable elements. Before the demonstration, we show a table of variables that will be used in this and the next sections in Table I.

The constructing process mainly includes *initialization*, *growth*, *connection*, and *termination*.

Initialization, in this paper, is a process of a small *NW* network. Assuming that the number of total vertices of the initial network is n. Each of them links to k neighbors, and has the probability p to link to others. Self-loops are avoided.

Growth is the key step of an evolving network, which consists of the vertex growing rate and arrival vertex connection. For each time t, we add $\lambda(t)$ vertices. In the interval $[t, t+\triangle t]$, the probability of number of new vertices is then

$$P\{N(t+\triangle t)-N(t)=k\} = \frac{[s(t+\triangle t)-s(t)]^k}{k!}e^{s(t+\triangle t)-s(t)},\ k=0,1,\ldots \quad (1)$$

where $s(t)=\int_0^t \lambda(s)ds$. Besides, for each vertex, we connect m edges to the m different vertices already present in the network, where m follows a log-normal distribution, i.e., the density of m is given by:

$$f(m)=\begin{cases}\frac{1}{m\sqrt{2\pi}\sigma}e^{-\frac{(\ln m-\mu)^2}{2\sigma^2}}, & x>0\\ 0, & x\leq 0.\end{cases} \quad (2)$$

Connection is simply linearly dependent on the degree of the target vertex for the benefit of the following derivation, $\phi(i)$, the probability of a connection to a vertex i, is denoted as

$$\phi(i)=\frac{k_i}{\sum_j k_j}. \quad (3)$$

Termination is controlled by time t, which directly affect the scale of networks.

C. Degree Distribution of Evolving Networks

In order to understand the subnormalization phenomenon based on the proposed model, we study the degree distribution as the main purpose of this paper. In this section, we let $t\to\infty$, then the small initial network has little effect on the degree distribution. Therefore, in the derivation, the initial network is ignored.

Given that the input rate of vertices is $\lambda(t)$, each new vertex links to m edges, $N(t)$ is independent of connections m. Then the expected value of the total degree is

$$\sum_j k_j = 2E[m]E[N(t)] = 2\mu s(t). \quad (4)$$

For a new vertex with m edges, one of which connects to the existing vertex i, the corresponding probability is

$$P=\binom{m}{1}[\phi(i)][1-\phi(i)]^{m-1}\approx\frac{mk_i}{2\mu s(t)} \quad (5)$$

where $\phi(i)$ relates to (3).

Obviously, for one unit time from t to $t+1$, the probability that the degree of vertex i increases one is approximately $\lambda(t)[(mk_i)/(2\mu s(t))]$. Then, we assume that the degree distribution of the vertex i $P_k(t+1)$ follows a master degree:

$$P_k(t+1)\approx\frac{\lambda(t)m(k-1)}{2\mu s(t)}P_{k-1}(t)+\left(1-\frac{\lambda(t)mk}{2\mu s(t)}\right)P_t(k) \quad (6)$$

then, we have

$$P_k(t+1)-P_k(t)=\frac{\lambda(t)m}{2\mu s(t)}\big[(k-1)P_{k-1}(t)-kP_t(k)\big]. \quad (7)$$

Note that the differences of t and k are both 1, and based on the definition of the partial derivative, we get

$$\frac{\partial P_k(t)}{\partial t} = \frac{-\lambda(t)m}{2\mu s(t)} \cdot \frac{\partial k P_k(t)}{\partial k}. \tag{8}$$

Then we multiply both sides of (8) by k, and integrate over k, that is

$$\int_0^{\infty} kdk \frac{\partial P_k(t)}{\partial t} = \frac{-\lambda(t)m}{2\mu s(t)} \int_0^{\infty} kdk \frac{\partial k P_k(t)}{\partial k}. \tag{9}$$

Considering that the definition of the expectant degree k_i and employing the integration by part, we can deduce that

$$\begin{aligned} k_i &\approx \int_0^{\infty} k P_k(t) dk \\ &= -\left\{ [k^2 P_k(i,t)]_0^{\infty} - \int_0^{\infty} k P_k(t) dk \right\} \\ &= -\int_0^{\infty} kd[kP_k(t)] = -\int_0^{\infty} kdk \frac{\partial [kP_k(t)]}{\partial k}. \end{aligned} \tag{10}$$

Thus, (9) equivalently denotes as

$$\frac{\partial k_i}{\partial t} = \frac{\lambda(t)m}{2\mu s(t)} \cdot k_i \tag{11}$$

the general solution is $k_i = C[s(t)]^{(m/2\mu)}$, where C is a constant.

Combining with the boundary condition $k_i(s^{-1}(i)) = m$, where $s^{-1}(i)$ is the inverse function of $s(t)$, which indicates the generation time of vertex i, then we have the solution

$$k_i = m\left[\frac{s(t)}{i}\right]^{\frac{m}{2\mu}}. \tag{12}$$

After k_i is obtained, we next solve the degree distribution of the evolving network. As previously stated, the random variable m follows a log-normal distribution denoted as $f(m)$, while the vertex i is randomly selected from the vertices of the evolving network. This means that i follows a uniform distribution on $[0, s(t)]$, denoted as $f(i)$, and their joint distribution is denoted as $f(i,m)$. Obviously, $f(m)$ and $f(i)$ are the marginal probability densities for joint probability density $f(i,m)$, and the connection variable is mutually independent of the selected vertex, which means that $f(i,m) = f(i)f(m)$.

Then, based on the definition of the degree distribution function, the joint degree distribution $P\{k_i(t) < k\}$ is derived as

$$\begin{aligned} P\{k_i(t) < k\} &= \iint\limits_{k_i(t)<k} f(i,m)didm \\ &= \iint\limits_{i>s(t)\left(\frac{m}{k}\right)^{\frac{2\mu}{m}}} f(i)f(m)didm \\ &= \int_0^k \int_{s(t)\left(\frac{m}{k}\right)^{\frac{2\mu}{m}}}^{s(t)} \frac{1}{m\sqrt{2\pi}\sigma s(t)} e^{-\frac{(\ln m-\mu)^2}{2\sigma^2}} didm \\ &= \int_0^k \frac{1-\left(\frac{m}{k}\right)^{\frac{2\mu}{m}}}{m\sqrt{2\pi}\sigma} e^{-\frac{(\ln m-\mu)^2}{2\sigma^2}} dm. \end{aligned} \tag{13}$$

For (13), to solve the derivative of the joint degree distribution, by applying the Leibniz integral rule, we yield

$$\begin{aligned} p(k) = P'\{k_i(t) < k\} &= \frac{d}{dk}\int_0^k \frac{1-\left(\frac{m}{k}\right)^{\frac{2\mu}{m}}}{m\sqrt{2\pi}\sigma} e^{-\frac{(\ln m-\mu)^2}{2\sigma^2}} dm \\ &= \frac{1-\left(\frac{k}{k}\right)^{\frac{2\mu}{m}}}{m\sqrt{2\pi}\sigma} e^{-\frac{(\ln m-\mu)^2}{2\sigma^2}} \cdot k' - \frac{1-\left(\frac{1}{k}\right)^{\frac{2\mu}{m}}}{m\sqrt{2\pi}\sigma} e^{-\frac{(\ln m-\mu)^2}{2\sigma^2}} \cdot 0' \\ &\quad + \int_0^k \frac{d}{dk} \frac{1-\left(\frac{m}{k}\right)^{\frac{2\mu}{m}}}{m\sqrt{2\pi}\sigma} e^{-\frac{(\ln m-\mu)^2}{2\sigma^2}} dm \\ &= \int_0^k \frac{-m^{\frac{2\mu}{m}}}{m\sqrt{2\pi}\sigma} e^{-\frac{(\ln m-\mu)^2}{2\sigma^2}} \frac{d}{dk} k^{-\frac{2\mu}{m}} dm \\ &= \frac{\frac{2\mu}{m}}{k^{\frac{2\mu}{m}+1}} \int_0^k \frac{m^{\frac{2\mu}{m}-1}}{\sqrt{2\pi}\sigma} e^{-\frac{(m-\mu)^2}{2\sigma^2}} dm. \end{aligned} \tag{14}$$

So far, the degree distribution is obtained. Obviously, if we let m in (14) be a constant, the distribution is reduced to a power-law distribution. That is to say, the power-law distribution is a specific case of this distribution and the internal mechanism for only those SF networks with constant connections. However, for most networks and other situations, variable elements always exist. So the scope of applicability of the obtained distribution is much broader. Therefore, we suggest that this distribution is a very promising direction to study adaptive or evolving networks.

More importantly, from the mathematical derivation, we discover that the immanent factor of the subnormalization phenomenon for real world is caused by evolving of network degree distribution. In detail, degree distribution for evolving networks evolves into a 2-D joint random variable, which consists of the selection of vertices following a uniform distribution and the connection of new vertices following a log-normal distribution. The Matthew effect relates to the boundary of the joint probability density function (PDF). Consequently, the determinants of a degree distribution are the selection rule (whether it is selected randomly or certainly), the number of newly added links (whether they are constants or variables following such as a log-normal distribution), and the connection mechanism (e.g., the Matthew effect), all directly affects the final degree distribution. We reveal this direction is more significant than the traditional view that the connection mechanism is the crucial factor for evolving networks.

Additionally, a more universal form is required for a wider application. We reveal that the exponential term of a degree distribution for an evolving network ($2\ \mu/\text{m}$) is traced to that one link has two degrees [see (4)]. If we break the limit of the network, and regard the value of m as the increment or decrement of income in economics. Then, one income can be spent on different places, which means the exponential term cannot only be ($2\ \mu/\text{m}$). In that sense, by setting the exponential term as a parameter γ, we then figure out the general form of this kind of distribution in the next section.

III. Subnormal Distributions for Evolving Networks and Their Statistical Properties

For the purpose of broadening the application field such as fitting the personal wealth distribution, the general rigorous definition of our proposed distribution and its statistical properties are required. Hence, in this section, we mainly focus on the definition of the subnormal distribution derived from an evolving network and displaying its statistical properties.

A. Definition and Derivation of Subnormal Distribution

For (13) and (14), theoretically, consider that the connection variable m as a pure log-normal distribution with the parameters λ and σ^2, while i as a uniform distribution, then the variable k referred to (12) is defined as a subnormal distribution in this paper. Above all, we present the detail definition of a general subnormal distribution by its PDF and cumulative distribution function (CDF) in mathematics.

Definition 1: A continuous random variable X is said to have a subnormal distribution with the parameter $\gamma > 0$, if its PDF is given by

$$f(x) = \begin{cases} \frac{\gamma}{x^{\gamma+1}} \int_0^x \frac{t^{\gamma-1}}{\sqrt{2\pi}\sigma} e^{-\frac{(\ln t-\mu)^2}{2\sigma^2}} dt, & x > 0 \\ 0, & x \leq 0 \end{cases} \tag{15}$$

or, equivalently, if its CDF is given by

$$F(x) = \begin{cases} \int_0^x \frac{1-(\frac{t}{x})^\gamma}{\sqrt{2\pi}\sigma t} e^{-\frac{(\ln t-\mu)^2}{2\sigma^2}} dt, & x > 0 \\ 0, & x \leq 0 \end{cases} \tag{16}$$

where t is a log-normal random variable with the parameters λ and σ^2.

At the first place, we prove that the function $f(x)$ indeed is a PDF.

Theorem 1: $f(x)$ in (15) is a PDF having the properties that $f(x) \geq 0$ and $\int_{-\infty}^{+\infty} f(x) = 1$.

Proof: Apparently, for $x < 0, f(x) = 0$, otherwise, $f(x) > 0$. Overall, $f(x) \geq 0$.

For all t, having $0 < t < x < +\infty$, we can exchange the order of integral, that is

$$\begin{aligned} \int_{-\infty}^{+\infty} f(x) &= \int_0^{+\infty} \int_0^x \frac{\frac{\gamma}{x^{\gamma+1}} t^{\gamma-1}}{\sqrt{2\pi}\sigma} e^{-\frac{(\ln t-\mu)^2}{2\sigma^2}} dtdx \\ &= \int_0^{+\infty} \int_t^{+\infty} \frac{\frac{\gamma}{x^{\gamma+1}} t^{\gamma-1}}{\sqrt{2\pi}\sigma} e^{-\frac{(\ln t-\mu)^2}{2\sigma^2}} dxdt \\ &= \int_0^{+\infty} \left[-\frac{1}{\gamma} x^{-\gamma}\right]_t^{+\infty} \frac{t^{\gamma-1}}{\sqrt{2\pi}\sigma} e^{-\frac{(\ln t-\mu)^2}{2\sigma^2}} dt \\ &= \int_0^{+\infty} \frac{1}{t\sqrt{2\pi}\sigma} e^{-\frac{(\ln t-\mu)^2}{2\sigma^2}} dt \end{aligned} \tag{17}$$

let $y = [(\ln t - \mu)/\sigma]$, that is $dy = (1/\sigma t)dt$, then by substitution y and dy into (17), we have

$$\int_{-\infty}^{+\infty} f(x) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = 1. \tag{18}$$

The results follow. ■

As stated above, a subnormal variable is a joint probability density of a uniform variable and a log-normal distribution, thus we have Theorem 2.

Theorem 2: Given random variables X and Y are mutually independent, and X follows a uniform distribution on $[0, a]$, Y follows a log-normal distribution with the parameters μ and σ, if:

$$Z = Y\left(\frac{a}{X}\right)^\gamma \tag{19}$$

then, the random variable Z follows a subnormal distribution with the parameter γ.

Proof: For $z \leq 0$

$$F_Z(z) = P\left\{Y\left(\frac{a}{X}\right)^\gamma \leq z\right\} = 0. \tag{20}$$

Otherwise, for $z > 0$, since X and Y are mutually independent, the joint probability density $f_Z(z)$ is the product of marginal probability densities $f_X(x)$ and $f_Y(y)$, then

$$\begin{aligned} F_Z(z) &= P\left\{Y\left(\frac{a}{X}\right)^\gamma \leq z\right\} = \iint_{y(\frac{a}{x})^\gamma \leq z} f_Z(z)dxdy \\ &= \iint_{y(\frac{a}{x})^\gamma \leq z} f(x,y)dxdy = \iint_{x > a(\frac{y}{z})^\gamma} f_X(x)f_Y(y)dxdy \\ &= \int_0^z \int_{a(\frac{y}{z})^\gamma}^a \frac{1}{\sqrt{2\pi}\sigma ax} e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}} dxdy \\ &= \int_0^z \frac{1-\left(\frac{x}{z}\right)^\gamma}{\sqrt{2\pi}\sigma x} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}} dx. \end{aligned} \tag{21}$$

Consider both (20) and (21), we have

$$F_Z(z) = \begin{cases} \int_0^z \frac{1-\left(\frac{x}{z}\right)^\gamma}{\sqrt{2\pi}\sigma x} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}} dz, & z > 0 \\ 0, & z \leq 0 \end{cases} \tag{22}$$

obviously, $F_Z(z)$ follows the CDF of a subnormal distribution.

Furthermore

$$f_Z(z) = F'_Z(z) = \begin{cases} \frac{\gamma}{z^{\gamma+1}} \int_0^z \frac{x^{\gamma-1}}{\sqrt{2\pi}\sigma} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}} dx, & z > 0 \\ 0, & z \leq 0 \end{cases} \tag{23}$$

indicates the PDF of a subnormal distribution.

In summary, the results follow. ■

Additionally, the integration of Definition 1 is difficult to calculate in practical situation. To address this issue, we can also use the error function also called Gaussian error function to denote (15) and (16) in Definition 1, which is presented in Theorem 3.

Theorem 3: The PDF of a subnormal distribution is given by

$$f(x) = \begin{cases} \frac{\gamma}{2x^{\gamma+1}} e^{\mu+\frac{1}{2}\gamma^2\sigma^2} \left[1 + \text{erf}\left(\frac{\ln x-\mu-\gamma\sigma^2}{\sqrt{2}\sigma}\right)\right], & x > 0 \\ 0, & x \leq 0 \end{cases} \tag{24}$$

and its CDF is given by

$$F(x) = \begin{cases} \frac{1}{2}\{\text{erf}(\frac{\ln x-\mu}{\sqrt{2}\sigma}) - \text{erf}[\frac{\ln x-\mu-(\gamma+1)\sigma^2}{\sqrt{2}\sigma}]\}, & x > 0 \\ 0, & x \leq 0 \end{cases} \tag{25}$$

where t is a log-normal random variable with the parameters λ and σ^2, and $\mathrm{erf}(x)$ is the Gaussian error function.

Proof: Consider Definition 1, for $x \le 0$, both $f(x)$ and $F(x)$ are equal to 0.

Otherwise, for $x > 0$, we set $y = [(\ln t - \mu - \gamma\sigma^2)/(\sqrt{2}\sigma)]$, and $dy = [1/(\sqrt{2}\sigma t)]dt$, noticing the integral range, then PDF can be expressed as

$$\begin{aligned} f(x) &= \frac{\gamma}{x^{\gamma+1}} \int_{-\infty}^{\frac{\ln x-\mu-\gamma\sigma^2}{\sqrt{2}\sigma}} \frac{e^{\gamma\left(\sqrt{2}\sigma y+\gamma\sigma^2+\mu\right)}}{\sqrt{\pi}\sigma} e^{-\frac{(\sqrt{2}y+\gamma\sigma)^2}{2}} dy \\ &= \frac{\gamma}{x^{\gamma+1}} e^{\mu+\frac{1}{2}\gamma^2\sigma^2} \int_{-\infty}^{\frac{\ln x-\mu-\gamma\sigma^2}{\sqrt{2}\sigma}} \frac{1}{\sqrt{\pi}} e^{-y^2} dy \\ &= \frac{\gamma}{x^{\gamma+1}} e^{\mu+\frac{1}{2}\gamma^2\sigma^2} \left(1 + \int_{0}^{\frac{\ln x-\mu-\gamma\sigma^2}{\sqrt{2}\sigma}} \frac{1}{\sqrt{\pi}} e^{-y^2} dy\right) \\ &= \frac{\gamma}{2x^{\gamma+1}} e^{\mu+\frac{1}{2}\gamma^2\sigma^2} \left[1 + \mathrm{erf}\left(\frac{\ln x - \mu - \gamma\sigma^2}{\sqrt{2}\sigma}\right)\right] \end{aligned} \tag{26}$$

where $\mathrm{erf}(x) = (2/\sqrt{\pi}) \int_0^x e^{-t^2} dt$.

Applying the same method, for $x > 0$, we solve CDF

$$F(x) = \frac{1}{2}\left\{\mathrm{erf}\left(\frac{\ln x - \mu}{\sqrt{2}\sigma}\right) - \mathrm{erf}\left[\frac{\ln x - \mu - (\gamma+1)\sigma^2}{\sqrt{2}\sigma}\right]\right\}. \tag{27}$$

The results follow. ■

Remark 1: The PDF can also be expressed as the complementary error function and standard normal CDF

$$\begin{aligned} f(x) &= \frac{\gamma}{x^{\gamma+1}} e^{\mu+\frac{1}{2}\gamma^2\sigma^2} \int_{\frac{\mu+\gamma\sigma^2-\ln x}{\sqrt{2}\sigma}}^{+\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} dy \\ &= \frac{\gamma}{2x^{\gamma+1}} e^{\mu+\frac{1}{2}\gamma^2\sigma^2} \mathrm{erfc}\left(\frac{\mu+\gamma\sigma^2-\ln x}{\sqrt{2}\sigma}\right) \\ &= \frac{\gamma}{x^{\gamma+1}} e^{\mu+\frac{1}{2}\gamma^2\sigma^2} \Phi\left(\frac{\ln x - \mu - \gamma\sigma^2}{\sigma}\right) \end{aligned} \tag{28}$$

where $\mathrm{erfc}(x) = (2/\sqrt{\pi}) \int_x^{\infty} e^{-t^2} dt$, and $\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-(t^2/2)} dt$.

Analogously, CDF can also be denoted as

$$\begin{aligned} F(x) &= \frac{1}{2}\left\{\mathrm{erfc}\left(\frac{\mu - \ln x}{\sqrt{2}\sigma}\right) - \mathrm{erfc}\left[\frac{\mu + (\gamma+1)\sigma^2 - \ln x}{\sqrt{2}\sigma}\right]\right\} \\ &= \Phi\left(\frac{\ln x - \mu}{\sigma}\right) - \Phi\left(\frac{\ln x - \mu - (\gamma+1)\sigma^2}{\sigma}\right). \end{aligned} \tag{29}$$

B. Some Statistical Properties of Subnormal Distribution

In practical situations, a distribution function is of the nonessential; instead some special properties are more useful. In this section, we provide some common statistical properties in numerals such as the expected value, variance, etc., and display their solving processes. The default of γ is nonzero in all derivations.

1) Expectation Value: In probability theory, the expectation value of a random variable is intuitively the long-run average value of repetitions of the experiment it represents. It is the weighted average of all possible values. Practically, if $Z = G(x, y)$ is a continuous random variable having a joint PDF $f(x, y)$, and $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |G(x, y)|f(x, y)dxdy < +\infty$ then the expectation value of Z is given by

$$E(Z) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(x, y)f(x, y)dxdy. \tag{30}$$

Then, we have the following theorem.

Theorem 4: The expectation value of a general subnormal variable with the parameter $\gamma \neq 1$ is $(\gamma/\gamma - 1)e^{\mu+(\sigma^2/2)}$.

Proof: A subnormal variable Z is jointed by mutually independent random variables, a uniform variable X distributed on $[0, a]$ and a log-normal variable Y distributed on $(0, +\infty)$, according to Theorem 2, and can be expressed as

$$z = y\left(\frac{a}{x}\right)^{\frac{1}{\gamma}}. \tag{31}$$

For $\gamma \neq 1$, we have

$$\begin{aligned} E(Z) &= \int_0^{+\infty}\int_0^{a} y\left(\frac{a}{x}\right)^{\frac{1}{\gamma}} f(x, y)dxdy \\ &= \int_0^{+\infty}\int_0^{a} y\left(\frac{a}{x}\right)^{\frac{1}{\gamma}} f(x)f(y)dxdy \\ &= \int_0^{+\infty}\int_0^{a} y\left(\frac{a}{x}\right)^{\frac{1}{\gamma}} \frac{1}{ay\sqrt{2\pi}\sigma} e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}} dxdy \\ &= \left[\frac{\gamma}{\gamma-1} x^{1-\frac{1}{\gamma}}\right]_0^a \cdot \int_0^{+\infty} \frac{a^{\frac{1}{\gamma}-1}}{\sqrt{2\pi}\sigma} e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}} dy \end{aligned} \tag{32}$$

let $t = [(\ln y - \mu)/\sigma]$, then

$$E(Z) = \frac{\gamma}{\gamma-1}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}+\sigma y+\mu} dt = \frac{\gamma}{\gamma-1} e^{\mu+\frac{\sigma^2}{2}}. \tag{33}$$

The result follows. ■

2) Variance: Additionally, we employ the variance to display the dispersion degree measuring how far a set of numbers is spread out. Specifically, for a variable X with the expectation value $E(X)$, then variance is given by $\mathrm{Var}(X) = E([X - E(X)]^2)$.

Then, we calculate the variance of a subnormal distribution.

Theorem 5: The variance of a general subnormal variable with the parameter $\gamma \neq 1$ and $\gamma \neq 2$ is $e^{2\mu+\sigma^2}[(\gamma/\gamma - 2)e^{\sigma^2} - (\gamma^2/[(\gamma-1)^2])]$.

Proof: The variance of a subnormal variable Z can be expressed as

$$\mathrm{Var}(Z) = E(Z^2) - [E(Z)]^2 \tag{34}$$

where, by utilizing (31)

$$\begin{aligned} E(Z^2) &= \int_0^{+\infty}\int_0^{a} y^2\left(\frac{a}{x}\right)^{\frac{2}{\gamma}} \frac{1}{ay\sqrt{2\pi}\sigma} e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}} dxdy \\ &= \frac{\gamma}{\gamma-2}\left[x^{1-\frac{2}{\gamma}}\right]_0^a \cdot \int_0^{+\infty} \frac{ya^{\frac{2}{\gamma}-1}}{\sqrt{2\pi}\sigma} e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}} dy \end{aligned} \tag{35}$$

here, we render $t = [(\ln y - \mu)/\sigma]$

$$\begin{aligned} E(Z^2) &= \frac{\gamma}{\gamma-2}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}+2\sigma t+2\mu}dt \\ &= \frac{\gamma}{\gamma-2}e^{2\mu+2\sigma^2}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(t-2\sigma)^2}{2}}dt \\ &= \frac{\gamma}{\gamma-2}e^{2\mu+2\sigma^2} \end{aligned} \tag{36}$$

and note that Theorem 4

$$[E(Z)]^2 = \frac{\gamma^2}{(\gamma-1)^2}e^{2\mu+\sigma^2}. \tag{37}$$

Then, (36) and (37) are substituted into (34), we have

$$\mathrm{Var}(Z) = e^{2\mu+\sigma^2}\left[\frac{\gamma}{\gamma-2}e^{\sigma^2} - \frac{\gamma^2}{(\gamma-1)^2}\right]. \tag{38}$$

The result follows. ∎

Remark 2: Apparently, if $\gamma = 1$ or $\gamma = 2$, the nonexistence of the variance follows. Furthermore, the variance should be non-negative, that is

$$e^{2\mu+\sigma^2}\left[\frac{\gamma}{\gamma-2}e^{\sigma^2} - \frac{\gamma^2}{(\gamma-1)^2}\right] \geq 0. \tag{39}$$

For $\gamma \neq 1$ or 2

$$e^{\sigma} \geq \frac{\gamma(\gamma-2)}{(\gamma-1)^2}. \tag{40}$$

Analogously

$$\sigma \geq \ln\frac{\gamma(\gamma-2)}{(\gamma-1)^2}. \tag{41}$$

Finally, we conclude that only if $\gamma \neq 1$ or 2 and $\sigma \geq \ln([\gamma(\gamma-2)]/[(\gamma-1)^2])$, the variance of the subnormal distribution will exist.

Remark 3: From Theorems 4 and 5, we obtain the relationship between the parameters μ, σ, and γ and the expectation value $E(X)$ and the variance $\mathrm{Var}(X)$. Specifically, μ is denoted as

$$\begin{aligned} \mu &= \ln\left[\frac{\gamma}{\gamma-1}E(X)\right] - \frac{1}{2}\ln\left\{\frac{\gamma(\gamma-2)}{(\gamma-1)^2}\left(1+\frac{\mathrm{Var}(X)}{[E(X)]^2}\right)\right\} \\ &= \ln\left[\frac{\gamma}{\gamma-1}E(X)\right] - \frac{1}{2}\sigma^2 \end{aligned} \tag{42}$$

and σ is

$$\sigma = \sqrt{\ln\{\frac{\gamma(\gamma-2)}{(\gamma-1)^2} + \frac{\gamma(\gamma-2)}{(\gamma-1)^2}\cdot\frac{\mathrm{Var}(X)}{[E(X)]^2}\}}. \tag{43}$$

From the derivation, we can see that the variance is possibly nonexistent. And if $\gamma \to \infty$, the variance is numerically equal to $e^{2\mu+\sigma^2}[e^{\sigma^2}-1]$, which is equivalent to the log-normal distribution. And we can learn from (42) and (43) that the values of both μ and σ are relatively low directing the variance to a low value, which agrees with the assumption in the process of derivation of a subnormal distribution.

3) Other Statistical Properties: For any real number k, the kth moment variable X is given by $E(X^k)$, thus we have the following theorem.

Theorem 6: The kth moment of a general subnormal variable with the parameter $\gamma \neq k$ is $[\gamma/(\gamma-k)]e^{k\mu+(1/2)k^2\sigma^2}$.

Proof: The kth moment of a general subnormal variable can be expressed as

$$\begin{aligned} E(Z^k) &= \int_0^{+\infty}\int_0^a y^k\left(\frac{a}{x}\right)^{\frac{k}{\gamma}}\frac{1}{ay\sqrt{2\pi}\sigma}e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}}dxdy \\ &= \frac{\gamma}{\gamma-k}\left[x^{1-\frac{k}{\gamma}}\right]_0^a\cdot\int_0^{+\infty}\frac{y^{k-1}a^{\frac{k}{\gamma}-1}}{\sqrt{2\pi}\sigma}e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}}dy \end{aligned} \tag{44}$$

let $t = [(\ln y - \mu)/\sigma]$, then

$$\begin{aligned} E(Z^k) &= \frac{\gamma}{\gamma-k}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}+k\sigma t+k\mu}dt \\ &= \frac{\gamma}{\gamma-k}e^{k\mu+\frac{1}{2}k^2\sigma^2}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(t-k\sigma)^2}{2}}dt \\ &= \frac{\gamma}{\gamma-k}e^{k\mu+\frac{1}{2}k^2\sigma^2}. \end{aligned} \tag{45}$$

The result follows. ∎

The arithmetic coefficient of variation $\mathrm{CV}(X)$ is the ratio $[\mathrm{SD}(X)/E(X)]$, where $\mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)}$.

Theorem 7: The $\mathrm{CV}(X)$ of a general subnormal variable with the parameter $\gamma \neq 1$ and $\gamma \neq 2$ is $[\gamma/(\gamma-k)]e^{k\mu+(1/2k^2)\sigma^2}$.

Proof: The $\mathrm{CV}(X)$ of a general subnormal variable can be expressed as

$$\begin{aligned} \mathrm{CV}(X) &= \frac{\mathrm{SD}(X)}{E(X)} = \frac{\sqrt{e^{2\mu+\sigma^2}\left[\frac{\gamma}{\gamma-2}e^{\sigma^2} - \frac{\gamma^2}{(\gamma-1)^2}\right]}}{\frac{\gamma}{\gamma-1}e^{\mu+\frac{\sigma^2}{2}}} \\ &= \sqrt{\frac{(r-1)^2}{r(r-2)}e^{\sigma^2}-1}. \end{aligned} \tag{46}$$

∎

The partial expectation (PE) value of a random variable X with respect to a threshold ξ is denoted as $PE(\xi) = \int_{\xi}^{\infty}xf(x)dx$.

Theorem 8: The PE value of variation $PE(X)$ of a general subnormal variable with the parameter $\gamma \neq 1$ is $[\gamma/(\gamma-1)]e^{\mu+[\sigma^2/2]}\Phi(\sigma - [(\ln\xi-\mu)/\sigma])$.

Proof: The $\mathrm{PE}(X)$ of a general subnormal variable can be expressed as

$$\begin{aligned} PE(X) &= \int_{\xi}^{\infty}xf(x)dx \\ &= \int_{\xi}^{+\infty}\int_0^x\frac{\frac{\gamma}{x^{\gamma}}t^{\gamma-1}}{\sqrt{2\pi}\sigma}e^{-\frac{(\ln t-\mu)^2}{2\sigma^2}}dtdx \\ &= \frac{\gamma}{\gamma-1}\int_{\xi}^{+\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(\ln t-\mu)^2}{2\sigma^2}}dt \end{aligned} \tag{47}$$

let $y = [(\ln t-\mu)/\sigma] - \sigma$, we have

$$\begin{aligned} PE(X) &= \frac{\gamma}{\gamma-1}\int_{\frac{\ln\xi-\mu}{\sigma}-\sigma}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}+\mu+\frac{\sigma^2}{2}}dy \\ &= \frac{\gamma}{\gamma-1}e^{\mu+\frac{\sigma^2}{2}}\int_{\frac{\ln\xi-\mu}{\sigma}-\sigma}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy \\ &= \frac{\gamma}{\gamma-1}e^{\mu+\frac{\sigma^2}{2}}\Phi\left(\sigma - \frac{\ln\xi-\mu}{\sigma}\right) \end{aligned} \tag{48}$$

where Φ is the standard normal CDF.

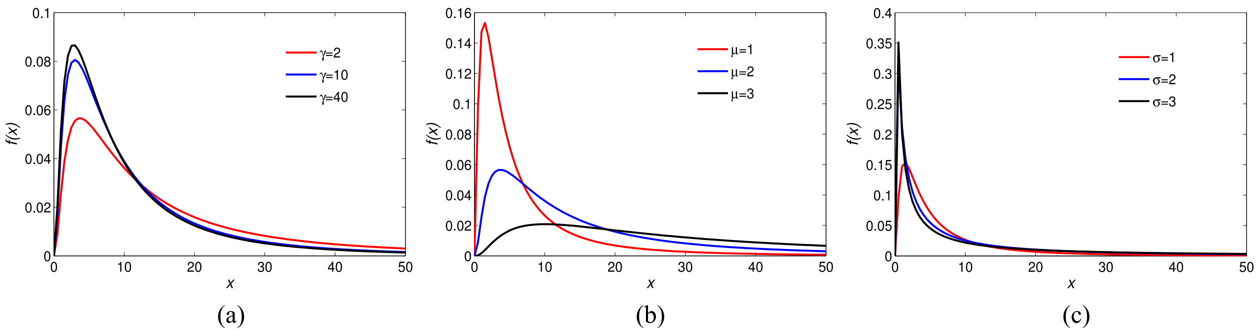

Fig. 1. Influence of different values of parameters on the curve of the subnormal distribution. The constant arguments are set as (a) $\mu = 2$ and $\sigma = 1$, (b) $\gamma = 2$ and $\sigma = 1$, and (c) $\gamma = 2$ and $\mu = 1$.

The result follows. ■

In addition, many other statistical properties, such as the characteristic function $E[e^{itX}]$, the moment generating function $E[e^{tX}]$ (which are easily proved divergent), mode, peak (which are without analytic solutions), etc., are omitted in this paper.

IV. Simulation and Analysis

In this section, we first provide theoretical simulations to analyze the influence of the different parameters γ, μ, and σ of Definition 1 on the curve of the subnormal distribution. Then, we carry out some simulations of fitting the subnormal distributions to the degree distributions of evolving network model. In the end, to fit those distributions in real world by our proposed distribution, we display the fitting simulations in scientific collaboration network as one kind of social networks and personal wealth as economic activities.

A. Parameter Analysis

To explore the influence of the parameters γ, μ, and σ on the curve of the subnormal distribution of (15), we let two of them be constants, and the other one deals with three distinct values. Then the corresponding plots are drawn. The results are illustrated in Fig. 1.

For the parameter γ, as shown in Fig. 1(a), the higher value makes the curve taller and thinner, which means the peak grows higher. However, a change from 2 to 10 is much more obvious than from 10 to 40. And the influence of γ on the mode (x of the peak) is inconspicuous. As a result, γ speeds up ascent rate before mode, and descent rate after the mode. This character is very similar to the exponent index of the power-law distribution denoting the slope in logarithmic coordinates.

We can see from Fig. 1(b), with the rise of the parameter μ, the curve becomes shorter and fatter. The change is very apparent, even μ rarely increases by 1. That is to say, μ is positively related to the mode, but has a visibly negative effect on the peak.

As the last parameter σ illustrated in Fig. 1(c), the higher value makes the curve taller and thinner. For the peak, the influence of σ is similar to γ. In other words, with the increase of σ, the peak rises, but very slow. Different from γ, σ is negatively related to the mode in the exponential, the higher value makes the mode much lower. Fig. 1(c) displays that the corresponding x move leftward.

In summary, γ and σ positively affects the peak of a subnormal distribution indicating the height of the curve, while μ does it negatively. Besides, μ has a positive influence on the mode indicating the location of the peak. Contrarily, σ has a negative influence, and γ has an inapparent influence on the mode. Furthermore, γ and σ have a positive relationship with the rate of rise and fall, the latter is more severe. Otherwise, μ has an obviously inverse relationship. With these relationships, we can determine the approximate curve shape of the required subnormal distribution.

B. Fitting Subnormal Distribution to Proposed Model

To fit these distributions, the subnormal distribution is required to be discretized, in other words, x can only be integers. To clearly compare the fitness of two distributions, we apply the Pearson product-moment correlation coefficient as the index. Specifically, for vectors of subnormal variables X and other variables, such as the degree distribution of evolving network Y, the correlation coefficient is denoted as

$$\rho_{X,Y} = \frac{E[(X - E(X))(Y - E(Y))]}{\mathrm{SD}(X)\mathrm{SD}(Y)}. \tag{49}$$

1) Fitting the Degree Distribution to Evolving Network: The modeling of the evolving network is referred to in Section II-B.

In the initialization, we build a *NW* small-world network with 20 vertices as the initial network, each connects to two neighbors and has a 50% chance to add a link to others. In the evolving process, for simplicity, we use a homogeneous Poisson process instead of a nonhomogeneous one since the growing is essentially independent of the degree distribution.

1) Set the values of input rate λ and termination time t.
2) Generate exponential distribution random values with λ, denoted as t_i, $i = \{1, 2, 3, \ldots\}$.
3) If the cumulative time $T_i \le t$, let $T_i = T_i + t_i$, else stop and output the temporal series.

Then we have the temporal series of arrival vertices. In the process of connection, we employ the *lognrnd* function in MATLAB to produce the number of connections. The result is rounded by *round* function, and the roulette algorithm is applied to simulate (3). As a result, a relatively sparse sample of a evolving network produced by $\lambda = 1$, $t = 100$, and $\mu = 1$, $\sigma = 1$ of log-normal is demonstrated in Fig. 4.

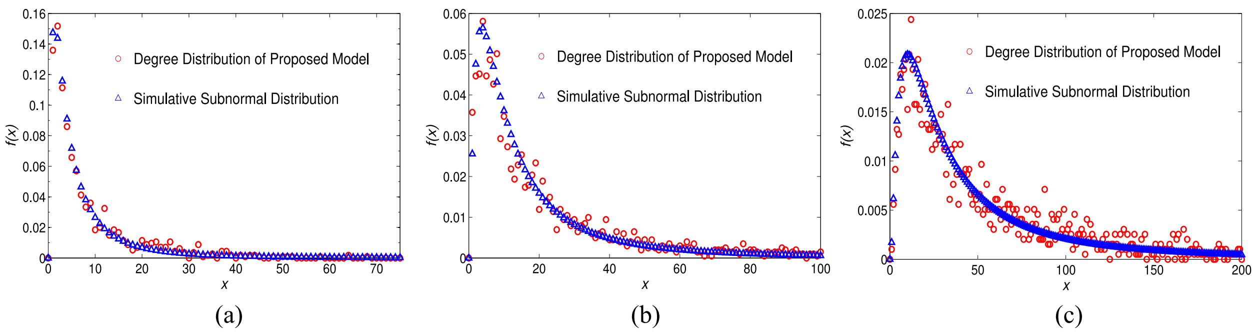

Fig. 2. Comparison of the degree distributions of evolving networks marked with red circles and their corresponding subnormal distributions marked with blue circles, all have that $\sigma = 1$ and $\gamma = 2$, but different μs. (a) $\mu = 1$. (b) $\mu = 2$. (c) $\mu = 3$.

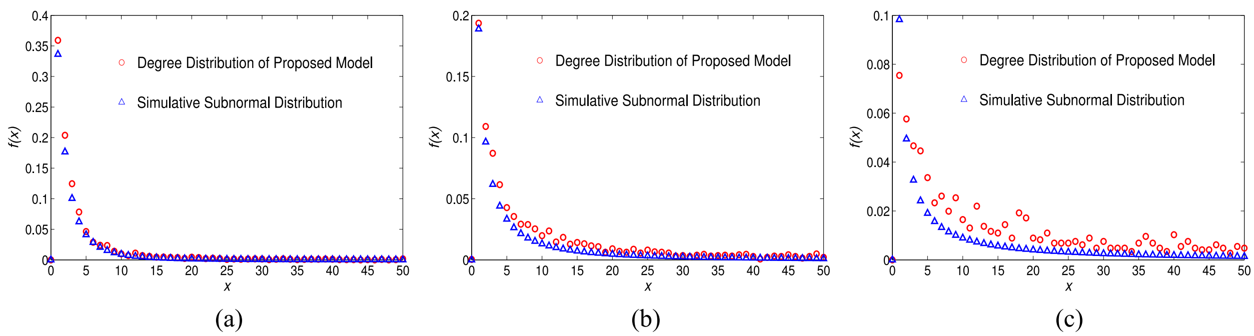

Fig. 3. Comparison of the degree distributions of evolving networks marked by red circles and their corresponding subnormal distributions marked by blue circles, all have that $\mu = 0$ and $\gamma = 2$, but different σs. (a) $\sigma = 1$. (b) $\sigma = 2$. (c) $\sigma = 3$.

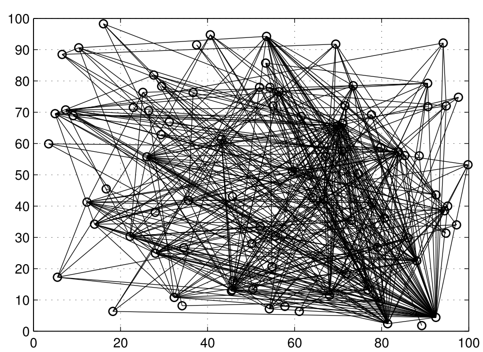

Fig. 4. Illustration of an evolving network produced by $\lambda = 1$, $t = 100$, and $\mu = 1$, $\sigma = 1$.

After an evolving network is obtained, we utilize the association matrix to record its degrees, and plot the corresponding degree distribution. Three evolving networks with different μs and σs are recorded. Then, with the same parameters, we also use Definition 1 to draw the distributions of the discrete subnormal variables in the same coordinate. The results are shown in Figs. 2 and 3. Since the distributions are obtained from networks, we let $\gamma \approx 2$.

In Fig. 2, different μs of the degree distributions and subnormal distributions are compared. To reduce the interruption of σ, we let it be the smallest integer 1. Since the tails of both distributions are extremely close, we take 75, 100, and 200 values of x for $\mu = 1$, 2, and 3 for clear illustration, respectively. By (49), the similarity of both distributions is calculated and listed in Table II(first row). From the results, we see that the correlation coefficient of two distributions are very high, all

TABLE II
CORRELATION COEFFICIENT OF THE DEGREE DISTRIBUTIONS OF EVOLVING NETWORKS AND THEIR CORRESPONDING SUBNORMAL DISTRIBUTIONS

Parameter \ Value	1	2	3
μ	0.9937	0.9810	0.9639
σ	0.9985	0.9908	0.9207

above 95%, implying that the degree distribution of evolving networks is highly similar to the subnormal distribution with the same parameters. And we also observe that the higher μ the lower the correlation coefficient, the deviation degree of both distributions becomes more obvious.

In Fig. 3, different σs of the degree distributions and the subnormal distributions are compared. To reduce the interruption of μ, we let it be the possibly smallest integer 0. For better display, 75, values of x for $\sigma = 1$, 2, and 3 are illustrated. By (49), the similarity of both distributions is calculated and listed in Table II(second row). The results also show a good evaluation of similarity of both distributions, all above 90%. However, the higher σ will raise the variance of connections, and consequently, the deviation degree is more apparent, leading to a lower correlation coefficient.

Above all, the subnormal distribution well fits the degree distribution of evolving networks, and the result is better for relatively low values of μ and σ.

C. Fitting Distribution of the Real World

Furthermore, to show the validity of the fitness of our proposed distribution in real world, we carry out the fitting

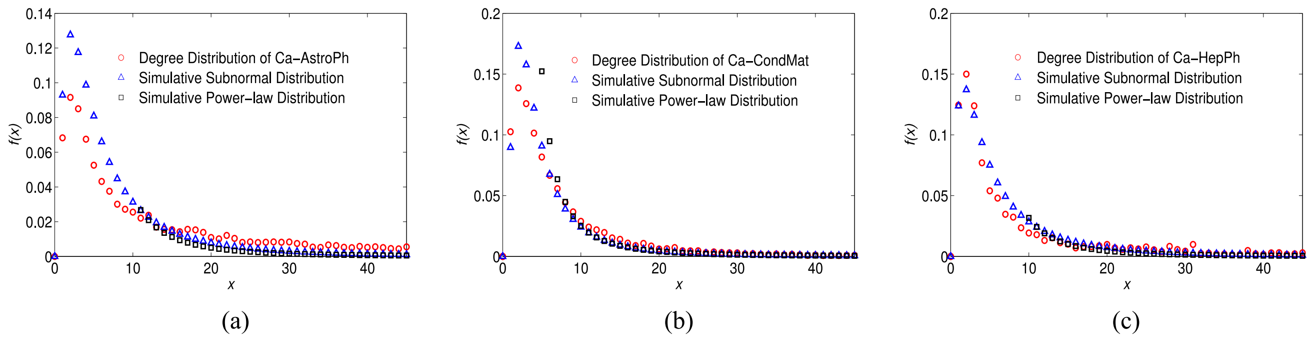

Fig. 5. Comparison of the degree distributions of three different collaboration networks marked by red circles and their corresponding subnormal distributions marked by blue circles and power-law distributions marked by black circles. Degree simulation for (a) Ca-AstroPh, (b) Ca-CondMat, and (c) Ca-HepPh.

simulations for those distributions of real networks in this section. They, respectively, aim at the scientific collaboration network degree distributions and personal wealth distributions.

1) Fitting Scientific Collaboration Network Degree Distributions: First, as one of the social networks for scientists, a collaboration network of arXiv astro physics (Ca-AstroPh) from the e-print arXiv which covers scientific collaborations between authors of papers submitted to astro physics category is fitted to the subnormal distribution [32]. And we also select another collaboration networks dealing with condense matter (Ca-CondMat) from the e-print arXiv and covers scientific collaborations between authors papers submitted to condense matter category to validate the fitness of our proposed distribution [32]. Additionally, as an alternative kind of the social networks for scientists, an arXiv high energy physics paper citation network (Ca-HepPh) from the e-print arXiv and covers all the citations is also simulated by our distribution [33], [34].

For the collaboration network of Ca-AstroPh and Ca-CondMat, if an author i co-authored a paper with author j, the network contains an undirected edge from i to j. The data of Ca-AstroPh covers papers in the period from January 1993 to April 2003. It begins within a few months of the inception of the arXiv, and thus represents essentially the complete history of its ASTRO-PH section. The number of nodes is 18 772, and of edges is 396 160. And The data of Ca-CondMat covers papers in the period from January 1993 to April 2003. It begins within a few months of the inception of the arXiv, and thus represents essentially the complete history of its COND-MAT section. The number of nodes is 23 133, and of edges is 186 936. For the citation network of Ca-HepPh, if a paper i cites paper j, the graph contains a directed edge from i to j. The data of Ca-HepPh covers 34 546 papers with 421 578 edges in the period from January 1993 to April 2003. It begins within a few months of the inception of the arXiv, and thus represents essentially the complete history of its HEP-PH section. From these networks, we can obtain their related association matrices, and calculate the degree distributions, shown in Fig. 5(a)–(c) as red scatters, respectively.

From the data, we can estimate the expectation value of degree of collaboration network of Ca-AstroPh, which is $E(\text{AstroPh}) = 21.1038 \approx 21$. Assuming that one subnormal distribution fits this network degree, as we know, the γ for networks approximatively equals to 2, then by employing Theorem 4, the relationship of the parameters μ and σ can be denoted as $\mu + [(\sigma^2)/2] = [(\log[E(\text{AstroPh})])/2] \approx 1.522$. Thus, we let $\mu = 1.15$ and $\sigma = 0.86$. In the same way, for collaboration network of Ca-CondMat, its expectation is $E(\text{CondMat}) = 8.0809 \approx 8$, so we let $\mu = 0.85$ and $\sigma = 0.62$. And for the last citation network Ca-HepPh, its expectation is $E(\text{CondMat}) = 19.7377 \approx 20$, similarly we let $\mu = 1.05$ and $\sigma = 0.94$. The illustrations of blue scatters of simulative subnormal distributions are, respectively, shown in Fig. 5(a)–(c).

In previous studies, the power-law distribution was most mostly utilized to describe social networks, similarly we also introduce a power-law distribution for comparison which is denoted as $f(x) = \gamma m^{\gamma} x^{-\gamma-1}$ $(x > \mu)$, where $\gamma = 2$. For Ca-AstroPh, the expectation value of power-law distribution, if the distribution fits the network degree, can be denoted as $2m = E(\text{AstroPh})$. Then, to simulate the collaboration network, we set the parameter $m \approx 11$. Equivalently, for Ca-CondMat, we let the parameter $m \approx 5$, and for Ca-HepPh, we let $m \approx 10$. Their black scatters as simulative power-law distributions are, respectively, displayed in Fig. 5(a)–(c).

Again, (49) is employed to calculate the correspondence of the subnormal, power-law distribution with the degree distribution of the collaboration network. The first 200 values are taken into calculation, but for a clear displaying, only the first 45 values are shown in Fig. 5. For Ca-AstroPh, the correlation coefficient with the subnormal distribution is 98.68%, while that with the power-law distribution is 76.31% only. For the benefit of power-law distribution, we also calculate the correspondence starting from the first value 11 of power-law distribution with partial distributions of collaboration network degree, the results are 97.45% for the subnormal distribution and 94.45% for the power-law distribution. For Ca-CondMat, the subnormal distribution is 98.83%, while the power-law distribution is 58.03% only, 99.42% and 94.58% if only the values beyond 5 are compared. And for Ca-HepPh, the subnormal distribution is 98.59%, while the power-law distribution is 75.93% only, 96.77% and 95.20% if only the values beyond 10 are compared. All the results are listed in Table III clearly.

Obviously, the subnormal distribution fits the collaboration network degree distribution much better in tendency and correlation than the power-law distribution, whether in full

TABLE III
CORRELATION COEFFICIENT OF THE DEGREE DISTRIBUTIONS OF SCIENTIFIC COLLABORATION NETWORKS WITH THEIR CORRESPONDING SUBNORMAL DISTRIBUTIONS AND POWER-LAW DISTRIBUTIONS

Distribution \ Network	AstroPh	CondMat	HepPh
Subnormal (all)	0.9868	0.9883	0.9859
Power-law (all)	0.7631	0.5803	0.7593

Distribution \ Network	AstroPh	CondMat	HepPh
Subnormal (partial)	0.9745	0.9942	0.9677
Power-law (partial)	0.9445	0.9458	0.9520

or partial plots. Actually, the power-law distribution only describes the tails, i.e., those authors having many collaborations, but ignores those ones having a few only. In the real world, from our perspective, the lower values that cannot be perfectly described by the power-law distributions require more attention, since they represent the majority and bring serious influence on the tendency as well as the statistical properties of the whole distribution. In particular, the highest collaboration number denoting the peak of the plot in the collaboration network, can be described as the most probable value of authors, and should be the priority research objects. Additionally, the power-law is monotonously decreasing, but social networks usually present a peak, indicating that the poorest are not the most, and the subnormal distribution perfectly fits that character.

2) Fitting the Personal Wealth Distribution: As mentioned above, the subnormal can also describe the inequality distribution in economy. Therefore, we try to fit the subnormal distributions to personal wealth distributions from different time periods and find that the results are irrelevant to time.

As we know, personal wealth distribution is not easy to measure, since people avoid reporting their total wealth routinely. And the statistical data are often quartered or more, which makes it difficult to seek for precise values of each wealth level. But when a person dies, all assets must be reported for the purpose of inheritance tax. Using these data and an adjustment procedure, the British tax agency, the Inland Revenue, reconstructed wealth distribution of the whole U.K. population. We mainly employ two data sets of total gross capital value obtained from their Web site during the different periods to verify the fitness of our distribution, one is from 2008 to 2010 (U.K.$_1$) [35], the other is from 2011 to 2013 (U.K.$_2$) [36]. These data divide the people into eight levels of net estate: 1) £0 to £50 000; 2) £50 000 to £100 000; 3) £100 000 to £200 000; 4) £200 000 to £300 000; 5) £300 000 to £500 000; 6) £500 000 to £1 000 000; 7) £1 000 000 to £2 000 000; and 8) over £2 000 000, we average the level as $\{0.25, 0.75, 1.5, 2.5, 4, 7.5, 10, 25\}*10^5$. For the data U.K.$_1$, the numbers of people for each level are {3053, 2382, 4207, 2515, 1682, 889, 224, 98}, For the alternative date U.K.$_2$, the numbers for each level are {3157, 2059, 3845, 2573, 1984, 1039, 289, 122}. Refer to the law of large numbers, we consider their frequencies as their probabilities, thus we can get

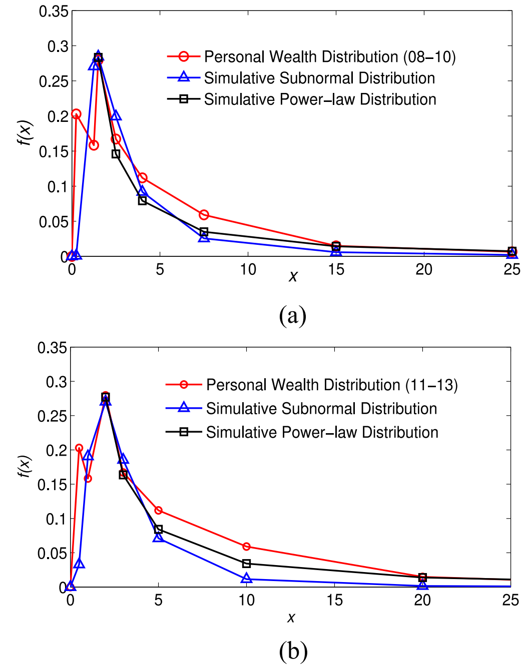

Fig. 6. Comparison of the personal wealth distribution marked by red circles and its corresponding subnormal distributions marked by blue triangles, power-law distribution marked by black squares. (a) U.K. Personal Wealth Statistics: 2008–2010. (b) U.K. Personal Wealth Statistics: 2011–2013.

the personal wealth distribution of U.K. during the two periods. Since the scatters are sparse, for clear illustration, we connect them as a plot [see the red plots in Fig. 6(a) and (b)].

As the data are sparse and too poor to calculate the expectation value, the previous method is not be able to evaluate the parameters of the fitting subnormal distribution. Thus, we test lots of parameters to present better results, one of them for U.K.$_1$ is $\gamma = 1.9$, $\mu = 0.6$, and $\sigma = 0.5$, and that for U.K.$_2$ is $\gamma = 1.9$, $\mu = 0.55$, and $\sigma = 0.55$ (see the blue plots of subnormal distributions in Fig. 6(a) and (b)].

For the power-law distribution, except the expectation value, we can also calculate the slope of the date in logarithm to obtain γ, and further m. The result for U.K.$_1$ is that $\gamma = 0.3$ and $m \approx 0.34$, for U.K.$_2$ is $\gamma = 0.3$ and $m \approx 0.35$. Then, the power-law distributions in black plots are illustrated in Fig. 6(a) and (b).

By (49), for U.K.$_1$, the result of correlation coefficient of subnormal distribution with wealth distribution is 80.53%, while the power-law distribution is 68.35% only, 99.24% and 97.55% if only the values beyond 2.5 are compared. And for U.K.$_2$, the result of subnormal distribution is 80.53%, while power-law is 68.35%, 99.24%, and 97.85% if only the values beyond 2.5 are compared. Once again, we display that the power-law only fits the tail, but the subnormal approximately fits the whole plot.

From Fig. 6, we can see that the U.K. wealth distributions during two different periods have no significant difference, the reason is that U.K. is a developed country, which means their personal distribution is stationary and free of time, and this perfectly follows our theory that the subnormal distribution is irrelevant to time once time is large enough. Besides, for the wealth, the part before peak of the distribution represents the lower-middle-classes, which play significant role in economic activities as well as stabilities, and should not be ignored as the power-law does. They hold a high percent in number of people, e.g., 64.07% lower-middle-classes for $U.K._1$ and 60.13% for $U.K._2$, and have a strong impact on the tendency of distribution. Therefore, as one of the best fit, the subnormal distribution can help to describe the personal wealth distribution and study its potential economic value.

It's worth noting that at the first three points make the tendency of plot decline, the underlying reason is the way they count. In detail, negative asset owners are included in 0 to 0.5 making the first positive value higher, but for both subnormal and power-law distribution negative values are ignored.

V. Conclusion

In this paper, we have presented a brand new distribution called *subnormal distribution* to simulate the distributions of the degree of evolving networks (such as SF networks), real networks (such as social networks and economic distribution) and other uneven distributions, which may help researchers study the properties of all these distributions. Essentially, from the derivation, we discover that the degree distribution is a 2-D joint probability density consisting of the selection of vertices that follows a uniform distribution and the connection of new vertices that in particular follows a log-normal distribution here, while the inequality of the Matthew effect relates to the boundary of the joint PDF. Actually, the connection may also be another distribution like the uniform distribution in some special cases, but in this paper we employ the log-normal distribution and obtain the subnormal distribution. In further work, we may extend the study to other joint PDFs as well.

We find that the subnormal distribution can also describe the wealth distribution, which can be explained by the network theory. The income is regarded as the new coming vertex, and the arrangement or consumption for this income is its connections to the network. The whole network represents the total wealth, which the degree each vertex represents the wealth of one individual. Apparently, the inequality of the Matthew effect influences the consumption of individuals. For example, people are more likely to buy goods of famous brands, and these firms are getting richer, which is highly similar with the connection process of evolving networks. Therefore, beyond the evolving networks, we speculate whether that the subnormal distribution can be universally employed to describe the distribution with inequality and growing which requires further studies.

Additionally, according to Gibrat's law, the size of a firm and its relative rate of growth are independent, which is also applied in evolving networks. One result of Gibrat's law is that processes characterized by Gibrat's law converge to a limiting distribution, which may be log-normal or power law, depending on more specific assumptions about the stochastic growth process. Furthermore, we precisely deduce that this kind of distribution based on the unequal growth follows that the rich tends to be richer while the poor is subnormal in this paper, which holds both characters of the log-normal and power-law distributions. For the special situation that the connection or income is constant, the subnormal distribution reduces to be power-law, and if the individual is constant and nonrandom, the distribution reduces to be log-normal. In that sense, we can also argue that the subnormal distribution is a combination of log-normal and power-law with the peak of the former and the tail of the latter. Above all, we agree with Gibrat's law that the income/connection is log-normally distributed, while the final wealth/degree follows a subnormal distribution.

However, we also have a dilemma on how to confirm the parameters of subnormal distribution. In this paper, we can only roughly determine the influence of γ, μ, and σ on the tendency of a subnormal curve, but cannot accurately deduce their attributes for a subnormal function. One possible solution is to solve the mode and median of the PDF, which greatly contributes to determining the function curve and is our goal for the next stage. The network model in this paper can be used to fit the degree distribution of the real network well, in addition to degree, there are many other criterions in networks such as the clustering coefficient, the average shortest paths, k shell, and so on, which are also worthy of study in the future.

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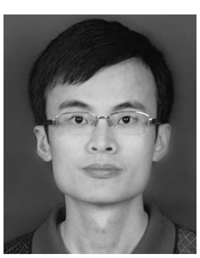

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