Chaos

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ABSTRACT

The preferential attachment of the Barabási–Albert model has been playing an important role in modeling practical complex networks. The preferential attachment mechanism describes the role of many real systems, which follows the characteristic "the rich get richer." However, there are some situations that are ignored by the preferential attachment mechanism, one of which is the existence of the limited resource. Vertices with the largest degree may not obtain new edges by the highest probability due to various factors, e.g., in social relationship networks, vertices with quite a lot of relationships may not connect to new vertices since their energy and resource are limited. Hence, the limit for degree growing is proposed in our new network model. We adjust the attachment rule in light of the population growth curve in biology, which considers both attraction and restriction of the degree. In addition, the unaware–aware–unaware opinion diffusion is studied on our proposed network. The celebrity effect is taken into consideration in the opinion diffusion process.

Since the scale-free network model was proposed, causing a sensation, network modeling has been studied by a lot of researchers. The purpose of studying network modeling is to properly build more appropriate network models for fitting practical networks. We establish an evolving network model considering limited resources for the degree of growth and propose the mechanism for network modeling in light of the population growth curve in biology. In addition, the UAU (Unaware–Aware–Unaware) opinion diffusion is investigated on the limited resource networks, where the celebrity effect is taken into consideration in the diffusion process. We extend the UAU diffusion model by adjusting the affected rate to a value depending on the degree value, in which a vertex is more likely to be aware influenced by its neighbor with a large degree. Through the Markov chain method, the diffusion threshold is analyzed. In various simulations, the degree distribution of our network is presented. The awareness fraction and the theoretical threshold based on different diffusion processes are also demonstrated.

I. INTRODUCTION

In recent years, the development of information technology has opened an avenue for complex network research. Network modeling and its dynamic propagation have never been valued by virtue of big data. Complex networks are applied for research in various fields, e.g., the biology protein network,[1] the transportation network,[2] and the power grid.[3] For better understanding of the structure and properties of real networks, a lot of network models were proposed to model and simulate lots of practical networks.

Watts and Strogatz first proposed the small-world networks featuring large clustering coefficient and low average path length.[4] Soon after, the BA scale-free network model was established by Barabási and Albert.[5] The connection mechanism in the BA model allows vertices with a larger degree to obtain more degrees with a higher probability, which is called preferential attachment. This connection mechanism explained the Pareto principle theoretically, and the BA scale-free model caused a sensation in the complex network; then, a great deal of work regarding network modeling emerged. Based on the small-world network, its topology properties were further studied by the mean-field method.[6] As an extension of scale-free networks, a local-world evolving network was proposed, which demonstrates a degree distribution between power-law and exponential scaling.[7] In addition to static networks, dynamic networks were established by lots of researchers. Adaptive networks arose and were applied in a large variety of fields.[8] A time-varying network defining the activity potential for each node[9] was proposed, and its further work was investigated in Ref. 10, where a continuous-time discrete-distribution method was established to investigate the epidemic spreading. Later, temporal networks were proposed, in which edges are not continuously active.[11] From a stochastic perspective, an evolving network was constructed by a novel mechanism applying the Poisson process.[12] The degree distribution of a network with the network size varying with time was further studied.[13] In addition to evolving network based on the stochastic process, the deletion mechanism was discussed specifically, which considers the heritable feature of node deleting.[14] These models consider the time-varying feature of networks. Additionally, by virtue of more access to real data, multilayer networks appeared, taking on a multiplex character of systems in real life.[15] An exponential random graph model showed that interest group influence reputations vary locally in multiplex networks.[16]

Based on various network models, dynamic studies like the epidemic spreading and opinion diffusion have been discussed on all sorts of networks. Epidemic spreading was studied on scale-free networks[17,18] and the researchers presented that the epidemic threshold was absent in scale-free networks.[19] Later, dynamics were also studied on layered networks,[20] temporal networks,[21] etc., investigating the impact of the network structure on dynamics. In addition, dynamic properties of epidemic propagation were also studied on weighted networks.[22] Recently, dynamics on multiplex networks have gained much attention, e.g., a novel epidemic model based on two-layered multiplex networks reveals the influence of preventive information on epidemic prevalence,[23] and studies on multiplex demonstrated that the link overlap facilitates the viability and mutual percolation.[24] Researchers also presented that the infected rate in multiplex networks is higher than that in isolated networks.[25] In addition to epidemic spreading, diffusion of opinion was also studied on the top of multilayer networks[26] and two-layer interconnected networks.[27] In order to fit situations in real life, reality factors were also studied in the opinion propagation. The community structure of social networks was considered, presenting the influence of the time-varying property of a modular structure on the information propagation.[28] Real-world collective phenomena were taken into consideration in opinion propagation by constructing a two-layered network to study the coevolution of opinion dynamics and decision-making.[29] Opinion leaders were considered in social contagion processes.[30] This pioneering work provides important insights into network modeling and dynamics on networks.

However, there are still crucial issues on unfitting real-life networks, e.g., ignoring the decreasing of networks, the disconnection of vertices, and the mechanism of connection. Though the BA model proposes a preferential attachment, which contributes to the heave-tail degree distribution, many networks in real life are not the same case as the BA attachment mechanism because of the limited resources. In other words, new vertices may not be connected to an existing vertex with the largest degree by the highest probability.

Based on the BA network model, in this paper, we establish a novel network model considering the limited resources for the vertex attachment mechanism. We take the resources limit into consideration and introduce a limited value to constrain the degree growing in networks. The expression of preferential attachment is adjusted, which includes the attraction and the resources limit of a vertex. The probability that a vertex receives an edge from a new vertex is decided by both the degree and the limit value. We also propose the disconnection mechanism that vertices break the edges with neighbors by a certain probability. In addition, we study the opinion diffusion on our proposed networks and investigate the influence of limit value on the propagation process.

The organization of this paper is as follows: in Sec. II, we display the construction of the resource-limit network. In Sec. III, we investigate the diffusion of opinions on this network. Simulations are carried out to demonstrate the validity of our models and theorems in Sec. IV. Conclusions and future work are given in Sec. V.

II. EVOLVING NETWORKS WITH LIMITED RESOURCES

Scale-free networks were proposed with the growth and the preferential attachment properties, which follows a power-law degree distribution. The mechanism of its preferential attachment and its degree distribution presents the Pareto principle, i.e., the rich get richer and the poor get poorer. While for many real networks, there is always a limit for growth and attachment. In other words, the degree of a vertex may not increase without limit, e.g., tutor–student networks where a tutor is unable to supervise too many students due to the limited personal energy and resources; therefore, it is not always the same case that students are more likely to connect to the tutor whose connections are the most. In fact, they are likely to connect to those tutors whose connection number is in a middle level. We next present our model in detail.

A. Preparation for modeling

Taking the resources limit for the growth of degree into consideration, in this section, we establish a novel model of evolving networks with the limited attachment mechanism. The probability that a newly coming vertex connects to an existing vertex i is not $k_i/\sum_j k_j$ any longer. In light of the curve of the population growth in biology, we adjust the expression of Π of the preferential attachment mechanism. Without natural resources limit, the population growth rate constantly increases with the population enlarging, whose growth curve manifests a "J" shape. Under the restriction of natural resources, the population growth rate increases at the beginning and then drops, and it stops growing until the growth rate becomes 0, in which case the growth curve presents an "S" shape. The mathematical expression of the "S" type curve is the differential equation that is $\frac{dN}{dt} = \frac{N(M-N)}{M}$, where N denotes the population size, M denotes the maximum size set for the population. The differential equation, which is called the logistic equation, depicts the growth rate varying with the population size when a limited size value is set to constrain the growth of the population. We can see that when the population size reaches M, the growth rate becomes 0. The growth rate reaches peak value when N is equal to half of the value of M. This growth rate expression gives the reason for the growth curve taking on an S shape. Based on this, the expression of preferential attachment, which describes the limit for the degree has a similar form to the logistic equation. In this case, vertices with middle degree values are more likely to obtain edges from a new vertex.

B. Modeling of network with limited resources

In this subsection, we present the mechanism of the evolving network with limited resources as follows. To propose our model, we mainly describe four parts that are the initial network, the growth of networks, the connection and disconnection, and the termination.

Initial network. There are a few m_0 vertices in the initial network, connecting to each other randomly.

Growth of network. A new vertex with m_1 edges comes to the network at each time step, which contributes to the growth of the network. m_1 edges of the new vertex will be connected to the existing vertex in the network.

Limit setting. A maximum value M is set for limiting the degree of a vertex. Vertices are called unsaturated when the degree is below M while when the degree is over M they are saturated.

Connection and disconnection. The newly coming vertex attaches to existing vertices (both saturated and unsaturated vertex) via its m_1 edges. The probability that a new vertex connects to the existing vertex i in the network depends on Π_i^+. Particularly, saturated vertices disconnect one of its neighbors by a certain probability decided by Π_i^-. The neighbors of a vertex are distributed uniformly; hence, we suppose that a saturated disconnects one of its neighbors randomly. Unsaturated vertices do not disconnect the edges with their neighbors.

Termination. We set time T large enough for the termination. Once time reaches T, the network ends up evolving.

Next, we focus on expressions Π_i^+ and Π_i^-. The logistic equation describing the population growth is introduced and gives important insight into the expression of Π_i^+ and Π_i^-. We replace the population size N with the degree of a vertex k_i in the logistic equation, then we have

$$\frac{dk_i}{dt} = r\frac{k_i(M-k_i)}{M}. \tag{1}$$

Equation (1) can be regarded as a description of the degree growth rate for vertex i in networks, which takes both degrees of a vertex and the limit for growing into consideration. From the expression, we can see that for unsaturated vertices with $k_i < M$, it is positive, while for saturated vertices with $k_i > M$, it leads to a negative value. In particular, it is equal to 0 when $k_i = M$. According to the original logistic equation in biology, the degrees of vertices with degrees less than the limit value M increase, the degrees of vertices with degrees equal to the limit value stop increasing, and those with degrees larger than the limit value decrease. However, there is a difference from the population growth that the vertices whose degree value larger than resources limit value in our network model can still receive edges from new vertices; thus, we take $r\frac{k_i(M-k_i)}{M}$ as the expression of Π_i^+ of all unsaturated vertices. For the saturated vertices, we take the minimum value of Π_i^+ among all the unsaturated vertices as all the saturated vertices' Π_i^i (where i denotes an unsaturated vertex hereby) to avoid a negative value. Then, Π_i^+ for unsaturated vertices is expressed as

$$\Pi_i^+ = r\frac{k_i(M-k_i)}{M} \tag{2}$$

and Π_j^+ for saturated vertices is denoted as

$$\Pi_i^+ = \min\left\{r\frac{k_i(M-k_i)}{M}\right\}. \tag{3}$$

We then consider Π_i^-, which is merely for saturated vertices since only saturated vertices have the probability to break edges initiatively and is expressed as $r\frac{k_i(M-k_i-1)}{M}$ to avoid the case that Π_i^- is 0 with $k_i = M$. Though Π_i^+ and Π_i^- are similar in form, they have different properties when considering the value of k_i. Particularly, in the range of $k_i > M$, Eq. (1) becomes negative and is a monotonously decreasing function, which indicates that the larger k_i ($k_i > M$) leads to the negatively smaller rate of disconnection.

Π_i^+ and Π_i^-, respectively, indicates the degree growth rate of vertices and the disconnection rate of unsaturated vertices. Analogous with the BA model, we use a probability to describe the connection and disconnection of vertices at each time step rather than utilize a certain growth rate. Therefore, we let Π_i^+ and Π_i^- normalized to transform them into the probability with the range of [0,1]. Then, we have the following definitions for P_i^+ and P_i^-.

Definition 1. In the proposed network, the connection probability Π_i^+ that vertex i obtains an edge from a new vertex is

$$P_i^+ = \frac{k_i(M-k_i)}{\sum_i k_i(M-k_i)}, \tag{4}$$

where k_i is the degree of vertex i, and M is the limited degree value for each vertex in networks.

According to Definition 1, saturated vertices have the least probability compared to unsaturated vertices, and unsaturated vertices whose degree reaches half of the limit value M have the largest probability to gain degrees through newly coming ones. This makes a difference to the BA network model in which the vertex with the largest degree have the highest probability to be connected to a new vertex.

Definition 2. For a saturated vertex i in the proposed network, the disconnection probability Π_i^- that it breaks the edge with one of its neighbors is

$$P_i^- = \frac{(k_j+1)(M-k_j-1)}{\sum_j (k_j+1)(M-k_j-1)}, \tag{5}$$

where $k_i \geq M$.

Definition 2 presents the disconnection probability of saturated vertices in networks. Distinct to those growth network models where the degree of vertices increases all the time, we consider the case that those saturated may break the connections to their neighbors due to the limited resources.

For better understanding, we hereby demonstrate an example of our proposed resource-limit network. The process of constructing a resource-limit network according to the mechanism in Sec. II B includes the network growth, the limit setting, connection and disconnection, and the termination in detail.

To build the network, the settings are $m_0 = 3$, $m = 1$, $M = 6$, and the termination $T = 8$. As shown in Fig. 1, at time $t = 0$, the initial network is a random graph with three vertices connecting to each other by a probability of 0.5. Specifically, there are two edges in total, a vertex with two degrees, and two vertices with one degree.

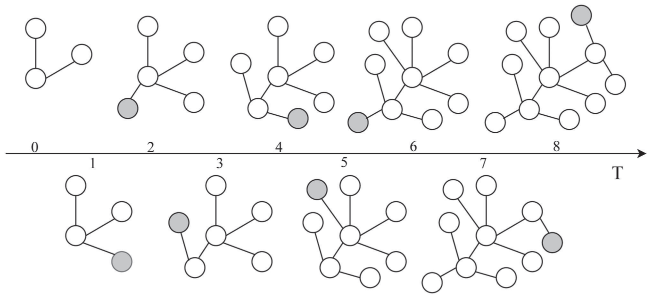


FIG. 1. An instance of an evolving network with limited resources: the figure displays a construction of an evolving network with limited resources according to our model. Termination time is set as $T = 8$, and for each time step, the gray vertex comes and attaches to an existing vertex by the probability expression we proposed.

For each time step, a new vertex comes and connects to existing vertices by the probability decided by $\Pi_i / \sum_j \Pi_j$. For instance, at $t = 2$, the newly coming vertex connects to the vertices with $k = 3$ and $k = 1$, respectively, by a probability of 0.286 and 0.179. At $t = 4$, the newly coming vertex connects to the vertices with $k = 4$, 2, and 1, respectively, by a probability of 0.222, 0.222, and 0.139, from which we can see that the probability to connect to a vertex with $k = 4$ and $k = 2$ is the same, presenting that vertices with the largest degree may not connect to the new vertex by largest probability due to the limited resources. After evolving some time, at $t = 8$, there is one vertex with $k = 5$, one vertex each with $k = 4$, one vertex with $k = 3$, and the others with $k = 1$. This instance helps understand the concrete construction of the limited resource network model.

C. Analyses for the transition probability of degree

In this subsection, we deduce the expression of probabilities that the degree of a vertex increases and decreases. The degree of an unsaturated or saturated vertex can increase via obtaining an edge from a new vertex. The degree of saturated vertices decreases by initiatively breaking edges with neighbors, and the degree of unsaturated vertices decreases through being disconnected passively by a saturated. Then, we have the following lemma.

Lemma 1. *The probability $P^i_{k,k+1}$ that degree of vertex i (both saturated and unsaturated vertices) increases by one is*

$$P^i_{k,k+1} = P^+_i \sum_{p=0}^{n} (1 - P^-_s)^p \left[P^-_s \left(1 - \frac{1}{\sum_j a_{sj}} \right) \right]^{n-p}, \tag{6}$$

the probability $P^u_{k,k-1}$ that the degree of unsaturated vertex u decreases by one is

$$P^u_{k,k-1} = 1 - \sum_{p=0}^{n} (1 - P^-_s)^p \left[P^-_s \left(1 - \frac{1}{\sum_j a_{sj}} \right) \right]^{n-p}, \tag{7}$$

and the probability $P^s_{k,k-1}$ that the degree of saturated vertex s decreases by one is

$$P^s_{k,k-1} = P^-_s + (1 - P^-_s) \sum_{p=0}^{n} (1 - P^-_s)^p \left[P^-_s \left(1 - \frac{1}{\sum_j a_{sj}} \right) \right]^{n-p}. \tag{8}$$

Proof. Suppose that the vertex j denotes a saturated neighbor of vertex i, and denote by A the adjacency matrix of the network. Then, the probability P^* of vertex i not being disconnected is

$$P^* = \prod_s \left[1 - P^-_s + P^-_s \left(1 - \frac{1}{\sum_j a_{sj}} \right) \right] = \prod_s \left(1 - \frac{1}{\sum_j a_{sj}} \right). \tag{9}$$

a_{sj} is the element of A denoting the connection between vertex s and its neighbor vertex j. The first term to the right of the equation is the probability that saturated neighbor j of vertex i is not involved in disconnecting at this time step. The second term is the probability that vertex j disconnects with its neighbors but vertex i is not chosen.

The probability of the degree of vertex i increasing by one is equal to the probability that vertex i being connected with a new vertex and not being disconnected by its neighbors, that is, P^+_i multiplied by P^*. Therefore, we get

$$P^i_{k,k+1} = P^+_i P^* = P^+_i \prod_s \left(1 - \frac{1}{\sum_j a_{sj}} \right). \tag{10}$$

Since unsaturated vertices lose degree passively by virtue of being disconnected with their saturated neighbors, the probability of the degree of an unsaturated vertex i decreasing is equal to the probability that vertex i not being disconnected with its neighbors. Thus, we have

$$P^u_{k,k-1} = 1 - P^* = 1 - \prod_s \left(1 - \frac{1}{\sum_j a_{sj}} \right). \tag{11}$$

The probability that the degree of a saturated vertex i descends is equal to the probability that vertex i disconnects with its neighbors actively or being disconnected with its saturated neighbors. The probability of vertex i actively disconnecting with its neighbors is Π^-_i, and the probability of vertex i not being broken edges by its neighbor is also P^*. Then, according to the total probability formula, we get

$$\begin{aligned} P^i_{k,k-1} &= P^-_i + (1 - P^-_i) P^* \\ &= P^-_i + (1 - P^-_i) \prod_s \left(1 - \frac{1}{\sum_j a_{sj}} \right). \end{aligned} \tag{12}$$

The results follow. □

Lemma 1 presents the probability of the degree increasing or decreasing in terms of a vertex from a microperspective, which can also be regarded as an expression for network evolving.

The degree of a vertex can be regarded as a discrete Markov chain since the future degree of a vertex is independent of the past degree and only depends on the present degree. Denote by the Markov chain of the degree of vertex i $\{k(t), t = 0, 1, 2, \ldots\}$, and the state of it is $0, 1, 2, \ldots$, which is infinite, since the degree of a vertex can grow infinitely. Particularly, $k(t) = 0$ indicates no disconnections of the vertex to any other vertices at time t. Suppose the probability that the degree is k at time step t is $P_k(t)$, based on Lemma 1, we obtain the transition matrix of vertex degree $k(t)$ as follows.

Definition 3. For each vertex, the one-step transition matrix H of the degree in the limited network is expressed as

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & 0 & \cdots & P_{k,k-1} & P_{k,k} & P_{k,k+1} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}. \quad (13)$$

According to Definition 3, we can obtain the expression of the degree distribution iteratively. Suppose that $P_k(t)$ is the probability that the degree of a unsaturated vertex is k at time t, and $p(t)$ is a vector whose elements are $P_k(t)$, $k = 0, 1, 2, \ldots$, then we get

$$p(t+1) = p(t)H = p(0)H^{t+1}. \quad (14)$$

According to Eq. (14), the degree distribution of vertices can be calculated iteratively by simulations. Above all, we demonstrate the construction of limited evolving networks in detail and perform analyses of the degree distribution by stochastic methods. Next, we discuss the opinion diffusion on our proposed network.

III. OPINION DIFFUSION ON RESOURCE-LIMIT NETWORK

In this section, we discuss the diffusion of the opinion on our proposed limited resource network. The limited resource network co-evolves together with opinions, i.e., the diffusion of opinion relies on the network, which is evolving all the time. The influence of the degree limit on the diffusion of opinions is mainly studied and by applying the micro-Markov chain method, we also analyze the threshold of the diffusion on the resource-limit network.

First, we briefly introduce the independent cascade model based on which we study the opinion diffusion on networks. Vertices are classified by aware (A) and unaware (U). Each vertex is affected by one of its neighbors, which is independent of its other neighbors. Each aware vertex affects its neighbors only once at a time step. In addition, for most cases in real life, information or opinion passes into silence after some time, we, thus, assume that there is a probability that an aware vertex becomes unaware again.

Then, we introduce the opinion diffusion model based on our constructed network. The Markov chain method is utilized to study the opinion propagation process.

A. Markov chain of UAU and analysis of threshold

In this subsection, we study the opinion diffusion on limited networks by the stochastic process theory. There are two states of a vertex, i.e., unawareness (U) and awareness (A), which can be taken as a Markov chain. Take the two states U and A as integers 0 and 1, which together compose the state space of the Markov chain. Specifically, that a vertex is under 0 state indicates that it is unaware while under 1 state indicates that it is aware. For an unsaturated vertex, it may be disconnected by its saturated neighbor according to the mechanism of the resource-limit network evolving. Hence, the structure of our proposed network has an influence on the opinion diffusion due to the disconnection inhibiting the diffusion between two vertices.

Primarily, we suppose that $P_i^0(t)$ is the probability that vertex i is unaware at time t, and $P_i^1(t)$ is the probability that vertex i is aware at time t. Then, we display the transition rates in the opinion diffusion. We consider two diffusion mechanisms where the affected rate β are different in the two mechanisms. The typical one is that for each vertex, the reviving rate of it transforming from 1 (A) state to 0 (U) state is α, and the affected rate of it transforming from 0 (U) state to 1 (A) state is β. Another one is that the rate of each vertex transforming from 1 (A) state to 0 (U) is also α, while the probability of it transforming from 0 (U) state to 1 (A) is related to the degree of its neighbors, expressed as

$$\beta_i = \beta_{min} \frac{\sum_j a_{ij}}{k_{min}}, \quad (15)$$

where k_{min} denotes the minimum degree of vertices in the network and β_{min} is the minimum affected rate that vertices with the minimum degree possess. According to Eq. (15), an unaware vertex is more easily affected by its neighbors with larger degree values than those neighbors with smaller degree values. This describes the variety of affecting abilities among vertices. A larger degree of vertices leads to a larger affecting ability, as a large degree indicates a large impact in real life. Then, we present the transitions between two states in Lemma 2.

Lemma 2. *Denoting by the adjacent matrix of the limited network A and taking* $\prod_{j=1}^{n} a_{ij} P_j^1 \beta_i$ *as* $r_i(t)$, *the probability of vertex i being aware* $P_i^1(t+1)$ *and unaware* $P_i^0(t+1)$ *at time* $t+1$ *is, respectively,*

$$\begin{aligned} P_i^1(t+1) &= P_i^1(t)(1-\alpha) + P_i^0(t) r_i(t), \\ P_i^0(t+1) &= P_i^1(t)\alpha + P_i^0(t)[1 - r_i(t)]. \end{aligned} \quad (16)$$

Proof. We assume that $r_i(t) = \prod_{j=1}^{n} a_{ij} P_j^1 \beta_i$ is the probability of vertex i becoming aware affected by its neighbors at time t. The probability that vertex i is aware at time $t+1$ is equal to the probability that vertex i is aware not reviving to be unaware at time t, denoted as $P_i^1(t)(1-\alpha)$, or vertex i is unaware becoming aware by its neighbors at time t, denoted as $P_i^0(t) r_i(t)$. Then, according to the total probability formula, the probability $P_i^1(t+1)$ that vertex i is aware at time $t+1$ can be expressed as

$$P_i^1(t+1) = P_i^1(t)(1-\alpha) + P_i^0(t) r_i(t). \quad (17)$$

Analogously, the probability that vertex i is unaware at time $t+1$ is equal to the probability that a vertex is unaware keeping its state next time at time t, denoted as $P_i^0(t)[1 - r_i(t)]$, or the vertex becomes aware at time t becoming unaware next time denoted as $P_i^1 \alpha$. Then, we have

$$P_i^0(t+1) = P_i^1(t)\alpha + P_i^0(t)[1 - r_i(t)]. \quad (18)$$

The results follow. □

Based on the above analyses, we next deduce the threshold for opinion diffusion. When the opinion spreads in the network, there is a threshold for the affected rate. We suppose that the reviving rate is α, and there is a value for the affected rate β_c, below which the probability of any vertex being aware P_i^1 becomes a quite small value tending to 0. In this case, the opinion will not spread throughout the network. Suppose that diag(x) indicates transforming a vector x into a diagonal matrix with the same element as x, and $\mathbf{1}$ is a vector with an element of 1. Then, we present the threshold in Theorem 1.

Theorem 1. *The threshold of the UAU opinion diffusion with the same vertex impact on the resource-limit network is*

$$\beta_c = \frac{\alpha}{\Lambda}, \tag{19}$$

where Λ is the largest eigenvalue of the adjacent matrix A. The threshold of the UAU opinion diffusion with the different vertex impact on the resource-limit network is

$$\beta_c = \frac{\alpha}{\Lambda'}, \tag{20}$$

where Λ is the largest eigenvalue of the matrix $A' = A^T \mathrm{diag}(A\mathbf{1})$.

Proof. As the propagation process is stationary, we have $P_i^1(t+1) = P_i^1(t) = P_i$. Suppose that P_i tends to ϵ_i where $\epsilon_i \ll 1$ when β tends to the threshold β_c, then we obtain

$$P_i^1(t+1) = P_i^1(t) = P_i = \epsilon_i, \tag{21}$$

and we also have

$$P_i^0(t+1) = P_i^0(t) = P_i = 1 - \epsilon_i. \tag{22}$$

Ignoring the second order terms of P_i^* and for the same receiving rate $\beta_i = \beta$, then we have

$$P_i^* = 1 - \beta \sum_j a_{ji}\epsilon_j. \tag{23}$$

Taking Eqs. (21)–(23) into the second formula of Eq. (16) in Theorem 2, it becomes

$$\epsilon_i = \epsilon_j(1-\alpha) + (1-\epsilon_i)\left[1 - \left(1 - \beta \sum_j a_{ji}\epsilon_j\right)\right]. \tag{24}$$

Neglecting the second order terms of ϵ, we get

$$\alpha\epsilon_i = \beta \sum_j a_{ji}\epsilon_j. \tag{25}$$

Processing Eq. (25), it transforms to be

$$\sum_j \left(a_{ji} - \frac{\alpha}{\beta}\delta_{ji}\right)\epsilon_j = 0, \tag{26}$$

where δ_{ji} is the Kronecker-delta. Then, the threshold is

$$\beta_c = \frac{\alpha}{\Lambda}, \tag{27}$$

where Λ is the largest eigenvalue of the adjacent matrix A.

In terms of the different affected rate, the probability of vertex i not being influenced P_i^* is

$$P_i^* = 1 - \beta \sum_i \left(a_{ji}\frac{\sum_j a_{ji}}{k_{min}}P_j\right). \tag{28}$$

Taking $(\epsilon_1, \dots, \epsilon_i, \dots, \epsilon_N)$ as vector ϵ where N is the dimension of adjacent matrix A, together with Eqs. (21) and (22), we obtain

$$k_{min}\alpha\epsilon = A^T \cdot \mathrm{diag}(A \cdot \mathbf{1})\epsilon. \tag{29}$$

Taking $A^T \mathrm{diag}(A \cdot \mathbf{1})$ as the matrix A', the threshold is

$$\beta_c = \frac{\alpha}{\Lambda'}, \tag{30}$$

where Λ' is the largest eigenvalue of the matrix A'. The result follows. □

The thresholds under the same and different affected rates were obtained via the Markov chain method, which both depend on the adjacent matrix of the proposed network. In the simulation, we will further illustrate the threshold opinion diffusion on our proposed network.

IV. SIMULATION

In this section, we display the simulation of our proposed network, the degree distribution, and the opinion diffusion on networks with different parameters is also investigated by presenting the level of awareness and the threshold. In detail, the degree distribution with different limit values M in our model, the stationary fraction of awareness with the same and different affected rates among individuals, and the threshold of the diffusion are demonstrated in the following.

A. Modeling of network

Primarily, we describe the construction of our network in simulation based on our model. In the simulation, we take time as discrete time steps. For each time step, a new vertex carried with m edges comes and connects to existing vertices in networks. An adjacent matrix is utilized to memorize the connections between vertices. The detailed procedures are as follows:

(1) Give the termination time N, which is also the size of the network, the initial number of vertices m_0, and the edge number m of a newly coming vertex. Calculate Π_i according to the degree and create an interval for each existing vertex to present the probability of being connected by new vertices.
(2) Generate a random number in the range [0,1] by *rand*() and find the corresponding vertex whose interval includes the generated number. Connect the new vertex to the chosen old vertex by turning 0 element to 1 of the adjacent matrix.
(3) Update the Π_i interval.
(4) Return (2) until achieving the termination.

Based on the above methods, we construct networks and demonstrate them in Fig. 2 where the settings are $N = 1000$, $m_0 = 3$, $m = 3$, and $M = 10, 40, 75$, respectively, for Figs. 2(a)–2(c). In addition, the size and color of nodes in plots represent the degree.

The bigger the size of the circle is, the largest degree value is. Yellow indicates a large degree while the other side blue indicates a small degree according to the color bar on the right of each subfigure. The measure of color presenting the degree is only within the same network. We can see that the small blue nodes are the most and large yellow nodes are the least in all three subfigures. The fraction of yellow and orange nodes of the network in Fig. 2(a) is larger than that in Fig. 2, and the fraction of yellow and orange nodes are the least in Fig. 2, which indicates that the degree is distributed almost equally with M while most unequally with $M = 75$. A strict limit for the network contributes to the equality of the degree distribution.

B. Degree distribution of limited resource network

As the degree distribution is the most important topology which reveals the character of networks, we construct our proposed limited resource networks with different limit values M and demonstrate degree distributions by simulation, presenting the influence of the attachment mechanism with the limit on the degree distribution of networks.

The initial number of vertices m_0 is set as 3, connected to each other randomly, and the number of edges m carried by a newly coming vertex is set as 2. According to our connection mechanism, we construct networks with different values for the limit value M. The time step is set as 10^3 for $M = 10$, 5×10^3 for $M = 20$, 20×10^3 for $M = 30$, 40×10^3 for $M = 40$, 80×10^3 for $M = 60$, and 120×10^3 for $M = 80$. The degree distributions of the resource-limit network with different values of M under the common coordinate are illustrated in Fig. 3.

Figures 3(a)–3(e), respectively, shows the degree distribution with limit value $M = 10$, 20, 30, 40, 60, and 80. The red line denotes the distribution of the power-law distribution with an exponent $\gamma = 3$, while the blue circle denotes the simulation results. As we can see, all of the distributions with a limit of the resource-limit are similar to the power-law distribution, while they have a milder slope than that of the given power-law distribution, especially in Figs. 3(a)–3(c). This may suggest that the limit value makes the degree distributed more equally compared on the whole with the original preferential attachment (PA) model. In addition, there existing degree value beyond the limit value M in Figs. 3(a) and 3(b), explaining that M is not the maximum degree value for degree stopping growing, it symbolizes a restriction for the degree growth under the resources limit.

For a better illustration, degree distributions of proposed networks under a logarithmic coordinate are also demonstrated. Figures 4(a)–4(e), respectively, illustrate the degree distribution of networks with limit value $M = 10$, 20, 30, 40, 60, 80 under double logarithmic coordinates. As we can see in Fig. 4, the degree distribution of the network with $M = 10$ in Fig. 4(a), in which its middle part obviously goes downward, is most different from the distribution of scale-free networks. The degree distribution in Fig. 4(e) is distributed as the theoretical distribution most closely. Degree distributions of networks with larger values of M are closer to the theoretical distribution of scale-free networks. However, subfigures in Fig. 4 show the difference between the degree distribution of the limited resource network with the power-law distribution. The tail of simulation results plot drops beneath the power-law distribution plot owing to the existence of limit value M, while simulation plots have a smaller slope than that of the power-law distribution plot except for the tail of plots. This indicates that the degree distribution with the limit value has a similar form as the power-law distribution on the whole, while most degree values are more homogenous than the power-law distribution, the fraction of quite large degree values (the tail of the distribution) is less than that of the power-law distribution. Additionally, simulation results suggest that with M increasing, degree distributions tend to the theoretical distribution. Particularly, when M approaches infinity, our proposed network becomes a scale-free network.

To obtain the exponent of the power-law distribution of simulation results, we estimate the parameters of their distribution formula via the linear regression method. The degree distribution of our proposed models follows the power-law distribution,

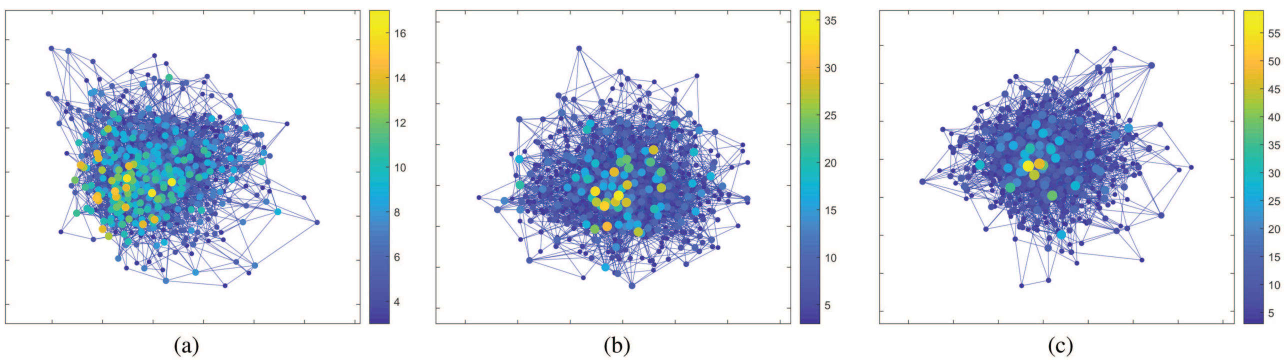


FIG. 2. Illustration of three resource-limit networks: the initial number of vertices is 3, and the network size is set as $N = 1000$. (a), (b) and (c), respectively, show the networks with the limit value $M = 10$, 40, and 75. The size and the color of nodes symbolize their degree values, where big and yellow circles indicate large degree values while small and blue circles indicate small degree values. We can see that the limit value M has an influence on the degree distribution. (a) Illustration of the network with $M = 10$. (b) Illustration of the network with $M = 40$. (c) Illustration of the network with $M = 75$.

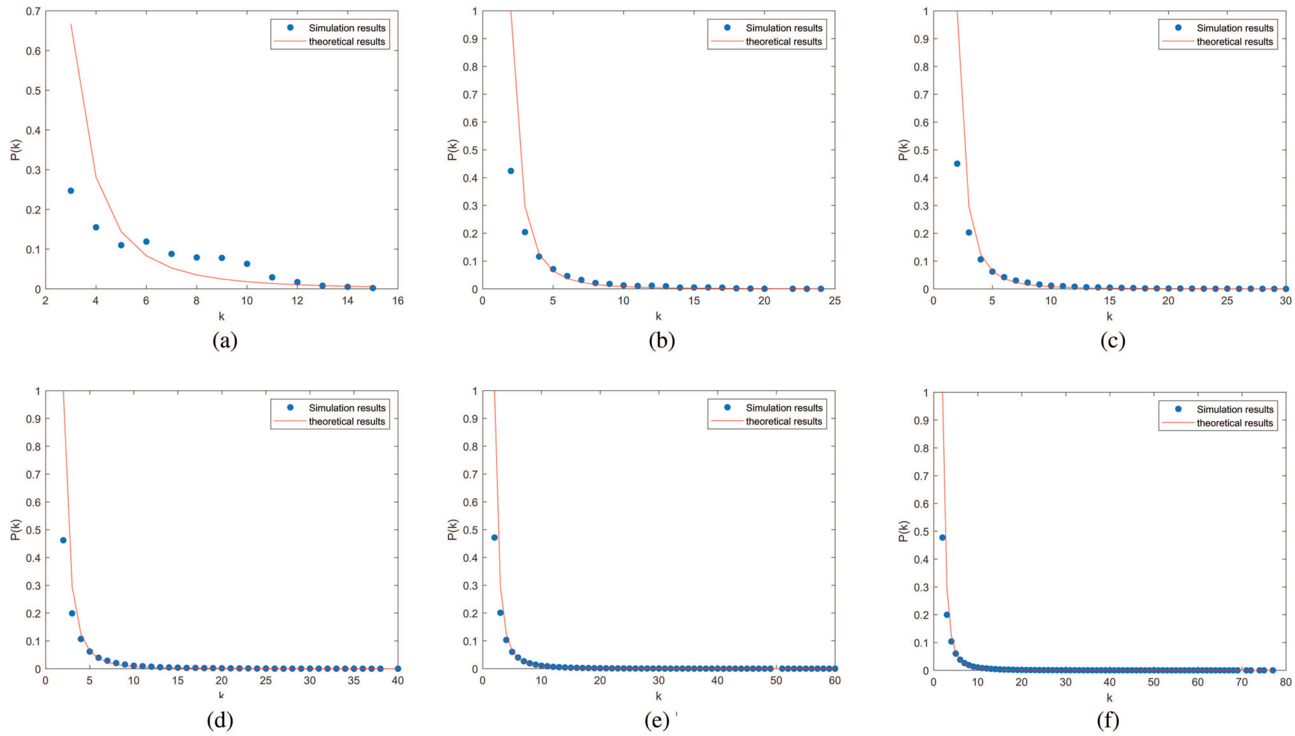


FIG. 3. The degree distribution with different limit values: (a)–(f) are, respectively, set with $M = 10$, 20, 30, 40, 60, and 80. The network size N is set as 1000. Red lines indicate power-law distributions with expression $2m^2k^{-3}$, and the blue circle plots indicate distributions of networks with resources limits. As we can see from (a)–(f), the degree distribution is getting close to the power-law distribution. (a) The degree distribution with the limit value $M = 10$. (b) The degree distribution with the limit value $M = 20$. (c) The degree distribution with the limit value $M = 30$. (d) The degree distribution with the limit value $M = 40$. (e) The degree distribution with the limit value $M = 60$. (f) The degree distribution with the limit value $M = 80$.

which can be mathematically denoted as $P(k) = Ck^{\gamma}$. We take both sides of the distribution equation as the logarithm and then we have

$$y = \gamma x + b, \tag{31}$$

where y is $\log P(k)$, x is $\log k$, and b is a parameter. Fitting the degree distribution via linear regression method in Python, the exponent γ and the parameter b are obtained. The results are illustrated in Table I in which we can see that the values of parameter γ which is the exponent of the power-law distribution is in the range of $[-3.08, -3.54]$. The parameter b is the y-intercept of the fitted regression line.

C. UAU opinion diffusion on networks

In this subsection, we simulate the process of opinion diffusion on our proposed networks. We present the density of aware vertices when the system is stationary with different affected rates and make a comparison between the theoretical threshold and the simulations results. In addition, propagation thresholds of the scale-free network based on the original PA model and our resource-limit model are displayed by calculating the theoretical threshold value based on concrete matrices.

First, we demonstrate the density of aware vertices in the proposed networks when the diffusion process is stationary. The underlying network is set as a constructed resource-limit network with a size $N = 1000$ and a limit value $M = 40$; the reviving rate α is constant, being 0.3, and the termination is set as $T = 3000$, large enough for the system to be stationary. In each experiment, we change the affected rate β, calculate the fraction of aware vertices, and then present the fraction of the awareness as a function of β. The stationary densities with different values of β are calculated as a mean value of over 200 realizations. As a result, Fig. 5 demonstrates two stationary densities of aware vertices as a function of affected rate in different diffusion processes. In Fig. 5(a), the value of affected rate β is the same among all the vertices, while in Fig. 5(b), the affected rate in the opinion diffusion is relevant to degree value, where the larger the degree k of the vertex is, the larger the affected rate β it has, following Eq. (15). The horizontal coordinate in Fig. 5(b) represents the median value of β.

We can see that both plots have an upward tendency as the affected rate increases, which indicates the density of aware vertices is in positive proportion with β. In detail, in both plots, the growth rate of densities of awareness becomes lower with β getting larger. In addition, the fraction of awareness in the diffusion with

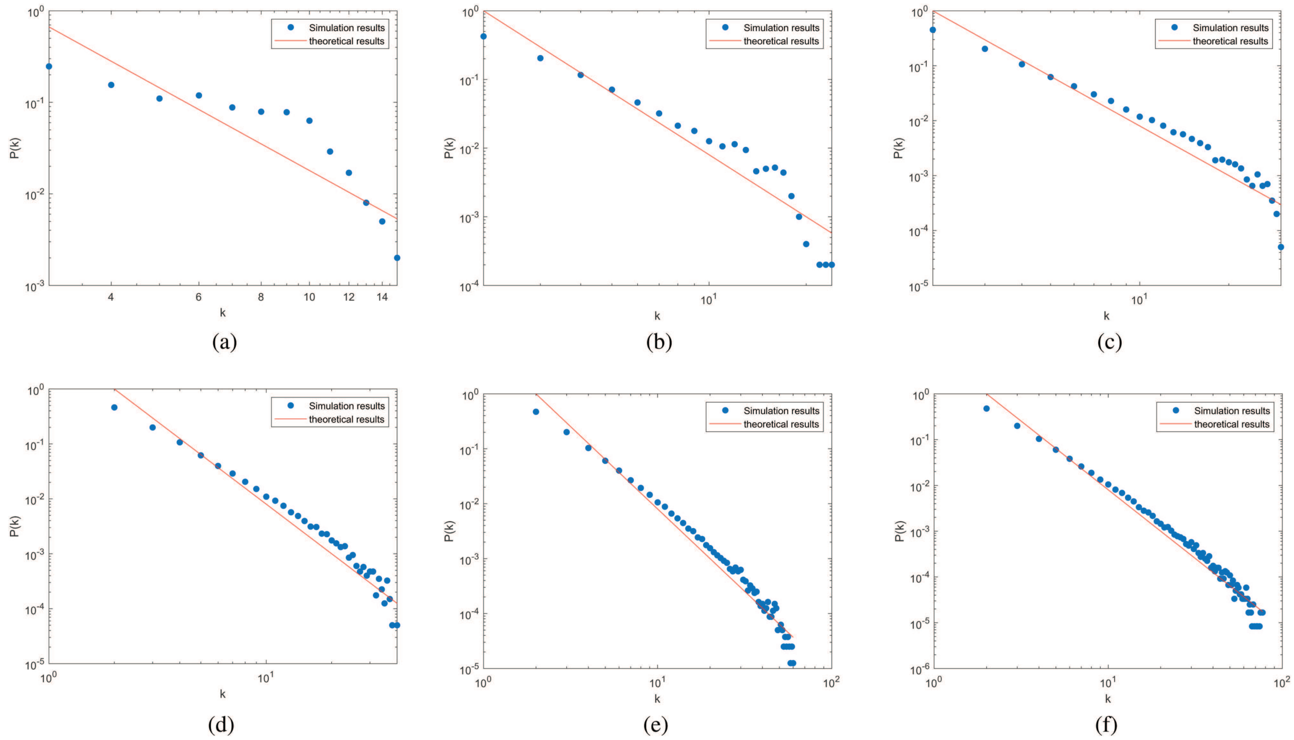


FIG. 4. The degree distribution under double log coordinate with different limit values: (a)–(f) is, respectively, set with $M = 10$, 20, 30, 40, 60, and 80. The network size N is set as 1000. Red lines indicate power-law distributions with expression $2m^2k^{-3}$, and the blue circle plots indicate distributions of networks with resources limit. The degree distributions from (a)–(f) are getting close to the power-law distribution under the double log coordinate. (a) The degree distribution with the limit value $M = 10$ under log coordinate. (b) The degree distribution with the limit value $M = 20$ under log coordinate. (c) The degree distribution with the limit value $M = 30$ under log coordinate. (d) The degree distribution with the limit value $M = 40$ under log coordinate. (e) The degree distribution with the limit value $M = 60$ under log coordinate. (f) The degree distribution with the limit value $M = 80$ under log coordinate.

k-dependent affected is larger than that with the same affected rate when β and median $\{\beta\}$ have the same value. In particular, when the affected rate is small, the awareness fraction with k-dependent β is much larger that with the same β among vertices, e.g., when β is 0.04 in Fig. 5(a), the corresponding density is 0.1, while the relevant density of median $\{\beta\} = 0.04$ in Fig. 5(b) is 0.25, which is two and a half times larger compared to Fig. 5(a). Additionally, the awareness fraction becomes over 0 when β is about 0.35 in Fig. 5(a), while in Fig. 5(b), the awareness fraction becomes over 0 when median $\{\beta\}$ is 0.01. In the underlying network with $M = 40$, nearly 40% vertices with the smallest degree have the smallest affected rate β. This declares that an increment in the affecting rate of vertices with

TABLE I. Parameters of degree distribution under logarithmic coordinate of networks with different limit values.

Parameter	Limit value M					
	10	20	30	40	60	80
γ	−2.54	−3.02	−2.83	−2.90	−3.05	−3.08
b	2.10	2.20	1.59	2.02	2.38	2.47

a large degree has a significant influence on the opinion diffusion where the stationary density of awareness will become larger. The results explain that in real life, a positive attitude of popular people or people with authority toward the opinion accelerates the opinion spreading, as they have a large affecting ability.

We next demonstrate the threshold of the opinion diffusion on our proposed resource-limit networks. We investigate theoretical thresholds on networks with different limit values of M and network size N. We first set M increasing from 10 to 85 by an increment of 5 each time, and N is set as 1000. Thresholds are calculated according to Eq. (27) in Sec. III A, and results are presented in Fig. 6(b), where we can see that the plot has a downward trend on the whole, illustrating that the threshold for β decreases with M increasing. In addition, when M is small (from 10 to 30), the decrease is apparent, while when $M > 30$, the dropping trend of the threshold becomes slow. The simulation results show that the rise of the limit value in our model reduces the threshold of opinion diffusion. In other words, the limit value can suppress the diffusion, while the absence of the limit facilitates the opinion spreading throughout the network.

In terms of the network size in Fig. 6(b), the plot also has a downward trend on the whole. The threshold for β descends

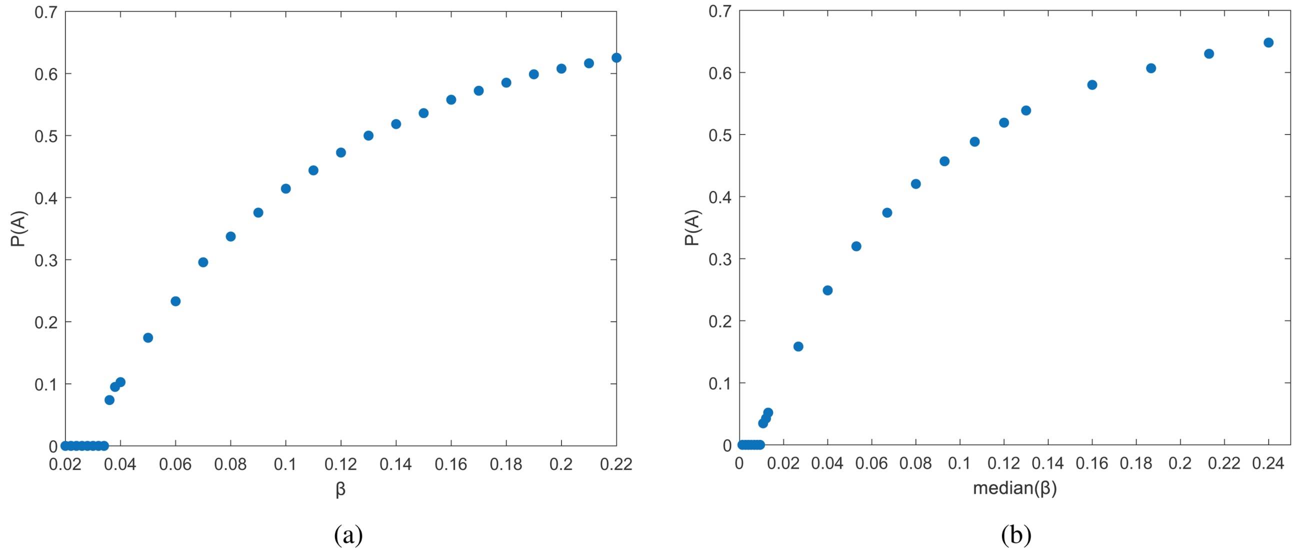


FIG. 5. Stationary fractions of awareness vertices as a function of the affected rate β: the underlying networks are set as $N = 1000$ and $K = 40$ in all subfigures. (a) presents the fraction of awareness with the same affected rate among vertices. (b) presents awareness fractions with k-dependent β, where a large k leads to a large β, and the horizontal coordinate indicates the median value of affected rates. The awareness fraction with k-dependent affected rate is higher than that with the same affected rate. (a) Stationary fractions of awareness under the same β among vertices. (b) Stationary fractions of awareness under degree-dependent β.

with the network size N reducing. Since the network is constructed randomly by simulations, the threshold is a not monotone decreasing with N. However, we can see that the threshold decreases when N increases on the whole, which indicates that the expansion of networks lowers the threshold of opinion diffusion on our proposed network. This corresponds to the threshold on scale-free networks with infinite size approaching 0.

Furthermore, we show the theoretical threshold for the minimum β of the degree-dependent opinion diffusion on resource-limit networks. The thresholds are also investigated on networks

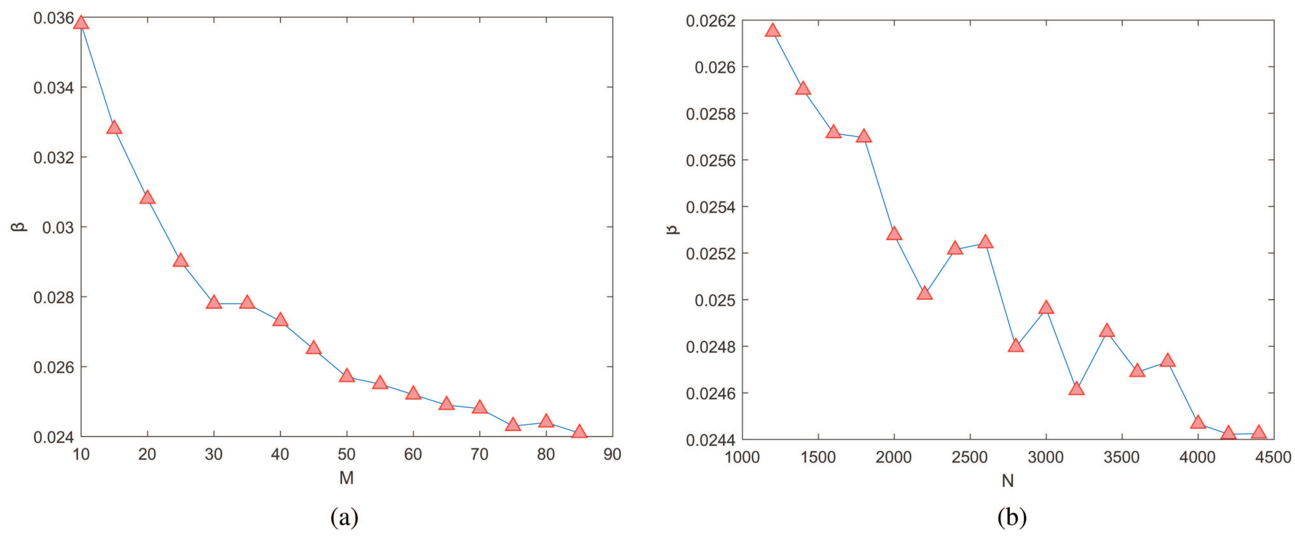
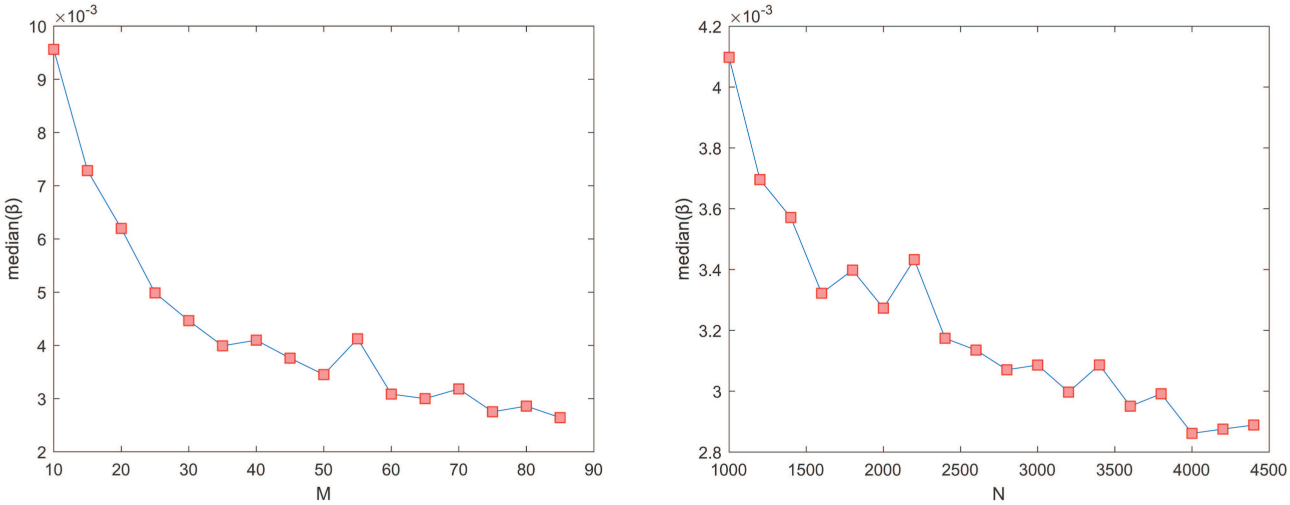


FIG. 6. Theoretical thresholds for the affected rate β with different network settings under the opinion diffusion with the same affected rate: the threshold for β is taken as a function of the limit value M in (a) and the network size N in (b). The threshold decreases as N and M increase. (a) Theoretical thresholds for β with different values of M. (b) Theoretical thresholds for β with different values of N.



(a) (b)

FIG. 7. Theoretical thresholds for the affected rate β with different network settings under the opinion diffusion with k-dependent affected rate: the threshold for median β is taken as a function of the limit value M in (a) and a function of the network size N in (b). The threshold decreases as N and M increase. (a) Theoretical thresholds for the median value of β with different values of M. (b) Theoretical thresholds for the median value of β with different values of N.

constructed by different limit values of M and the network size N in Fig. 7, where the threshold of minimum β is taken as a function of M and N, respectively. In Fig. 7(a), M is set from 10 to 85 by an increment of 5 each time, N is set as 1000, and the threshold generally descends with M increasing. This indicates that in the diffusion with degree-dependent affected rate, the rise of the limit value also reduces the threshold. In Fig. 7(b), N is set from 1000 to 4400 by an increment of 200 each time, M is set as 40, the threshold decreases with N increasing. This demonstrated that the expansion of networks lowers the threshold. In conclusion, the influence of network properties on the diffusion threshold is the same under two diffusion processes as shown in Figs. 6 and 7, while the degree-dependent affected rate facilitates the opinion spreading as shown in Fig. 5.

V. CONCLUSIONS AND OUTLOOK

In this paper, we propose an evolving network model considering limited resources for the degree growth and propose the mechanism for network modeling in light of the population growth in biology. The model can describe those networks and systems with limited resources, which restrains the growth of the degree. In the model, we analyze the growth of degree via the Markov chain. In addition, the UAU opinion diffusion is investigated on the limited resource networks. We take the celebrity effect into consideration in the diffusion process. The UAU diffusion model is improved by adjusting the affected rate to a value depending on the degree value, in which a vertex is more likely to be aware, influenced by its neighbor with a large degree. Through the Markov chain method, the diffusion threshold is analyzed. In various simulations, the degree distribution of our network is presented, suggesting a similar but not the same distribution as power-law distribution. The awareness fraction and the theoretical threshold based on different diffusion processes are also demonstrated.

Nevertheless, there are still some issues to be further addressed. For heterogeneous networks, it is complicated to give the concrete expression to generalize the increments of the number of individuals varying with time. Analytical solutions for the limited probability of nonhomogeneous Markov Chain have been a tough problem all the time. In addition, for practical applications, there is a requirement to apply our population model to fit realistic data. These issues will be further studied in our future work.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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