IEEE TRANSACTIONS ON

CYBERNETICS

A PUBLICATION OF THE IEEE SYSTEMS, MAN, AND CYBERNETICS SOCIETY

Indexed in PubMed® and MEDLINE®, products of the United States National Library of Medicine

MAY 2016 VOLUME 46 NUMBER 5 ITCEB8 (ISSN 2168-2267)

REGULAR PAPERS

TECHNICAL CORRESPONDENCE

Evolving Scale-Free Networks by Poisson Process: Modeling and Degree Distribution

Minyu Feng, Hong Qu, *Member, IEEE*, Zhang Yi, *Senior Member, IEEE*, Xiurui Xie, and Jürgen Kurths

***Abstract*—Since the great mathematician Leonhard Euler initiated the study of graph theory, the network has been one of the most significant research subject in multidisciplinary. In recent years, the proposition of the small-world and scale-free properties of complex networks in statistical physics made the network science intriguing again for many researchers. One of the challenges of the network science is to propose rational models for complex networks. In this paper, in order to reveal the influence of the vertex generating mechanism of complex networks, we propose three novel models based on the homogeneous Poisson, nonhomogeneous Poisson and birth death process, respectively, which can be regarded as typical scale-free networks and utilized to simulate practical networks. The degree distribution and exponent are analyzed and explained in mathematics by different approaches. In the simulation, we display the modeling process, the degree distribution of empirical data by statistical methods, and reliability of proposed networks, results show our models follow the features of typical complex networks. Finally, some future challenges for complex systems are discussed.**

***Index Terms*—Complex network modeling, degree distribution, degree exponent, Poisson process, power-law distribution, reliability.**

I. Introduction

WITH rapid development of Internet, information technology has become the focus of people's daily lives, and ushered the mankind into the age of complex networks. Network science based on information technology is now a significant issue and a useful tool to man's activities of production. As we know, many practical networks such as communication, traffic, social, and biological networks are regarded as complex systems with the same properties like small-world, scale-free, etc. However, practical networks are difficult to study, the key issue of network science is to create a virtual complex network model, which follows similar topology characteristics of practical networks.

In the late 20th century, two significant small-world models were proposed by Watts and Strogatz [1] and Newman and Watts [2]. They discovered that the practical networks are neither regular networks nor random graphs and should fall in between them, and realized them by randomly reconnecting and adding edges. Significantly, their models achieved the goal of a low distance between any two vertices and first simulates the small-world characteristic, the domination of ER random graph proposed by Erdös and Rényi [30] in network science is finally changed. Both ER random graph and WS small-world network have the same property that their degree distributions follow a Poisson distribution, having a peak value in the average degree that considered as exponential networks. However, the recent researches reveal that, most practical networks follow power-law distributions which are commonly called scale-free networks. Barabási and Albert [3] are the pioneer researchers to study the issue, and they proposed the famous BA scale-free network model. They take the growth and preferential attachment of practical networks into consideration, and successfully construct a network with a power-law distribution. Later, many methods such as the mean-field [4], rate-equation [5], and master-equation [6] were proposed to prove the power-law character of scale-free networks. In addition, there are many variants of BA model, such as temporal networks [7], spatial networks [8], aging networks [9], networks with accelerating growing [10], static networks [11], hierarchical networks [12], etc. Apart from BA network, there are also many other studies based on complex dynamics networks [13]–[20].

However, most scale-free networks consider their generation rates of vertices and number of connections as constants, but in practical situation, all variables keep changing over time. If so, are these networks still independent of time? In order to address this issue, we propose a novel scale-free network based on the Poisson process, which is an evolution of the BA scale-free network. Our model starts with a small group that highly gathered and connected to each other, and in the processing of modeling, we employ the homogeneous Poisson process to simulate the generation of vertices, and Gaussian distribution to produce the connection number. The dynamic method in physical and probabilistic approach in mathematic are utilized to analyze the degree distribution

Manuscript received November 21, 2014; revised February 5, 2015; accepted April 9, 2015. Date of publication May 4, 2015; date of current version April 13, 2016. This work was supported by the National Science Foundation of China under Grant 61432012 and Grant 61273308. This paper was recommended by Associate Editor J. Cao.

M. Feng and X. Xie are with the School of Computer Science and Engineering, University of Electronic Science and Technology of China, Chengdu 610054, China (e-mail: fmy1987@sina.com).

H. Qu is with the School of Computer Science and Engineering, University of Electronic Science and Technology of China, Chengdu 610054, China, also with the Potsdam Institute for Climate Impact Research, Potsdam 14473, Germany, and also with the Department of Physics, Humboldt University, Berlin 12489, Germany (e-mail: hongqu@uestc.edu.cn).

Z. Yi is with the College of Computer Science, Sichuan University, Chengdu 610065, China (e-mail: zhangyi@scu.edu.cn).

J. Kurths is with the Potsdam Institute for Climate Impact Research, Potsdam 14473, Germany, also with the Department of Physics, Humboldt University, Berlin 12489, Germany, and also with the Institute for Complex System and Mathematical Biology, University of Aberdeen, Aberdeen AB24 3UE, U.K. (e-mail: kurths@pik-potsdam.de).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TCYB.2015.2424425

and its relationship with time. In addition, assuming that the generation rate changes with time and the existing individuals keep dying, we extend this model to the nonhomogeneous Poisson and birth and death situation, which are more close to practical networks, with topologies discussed. In the view of application, these networks with variable generation rates can be utilized to simulate the practical situation which is difficult to obtain the data, and regarded as underlying models to solve issues like search, synchronization, etc. In the simulation, we demonstrate detailed construction processes of three networks, apply statistical approach to obtain the degree distribution and exponent from empirical data, and measure the reliability of these networks under deliberate and random attack.

This paper is organized as follows. A detailed presentation and analysis of the homogeneous Poisson network (HPN) is given in Section II. Introduction and analysis of nonhomogeneous Poisson network (NHPN) and birth and death network (BDN) are presented in Section III. Simulations are carried out in Section IV to demonstrate these networks follow power-law and are scale-free. Finally, the conclusion is drawn in Section V.

II. Scale-Free Network Based on Homogeneous Poisson Process

It is well known that the BA scale-free network is one of the first models proposed the growth characteristic. Both BA scale-free network and other similar networks simply consider that one vertex or individual is connected to the existing network per unit time. However, for real networks, the generation of vertices or individuals follows certain rules. For example, the population network does not generate one individual each time, instead, the growth rate changes all the time and is decided by many factors such as wars, economies, etc. Consequently, a common property of these networks is that the generation possesses a certain rate, which can be supposed as a counting process.

Besides, a BA scale-free network obtains permanent m edges for each vertex added in, but in real situation, connections of different vertices are very mutable. For example, in the network of the research reference, once a high quality paper such as emergence of scaling in random networks by Barabási and Albert [3] is published, many related references will emerge in a short time. Consequently, the number of connections is influenced by the vertex's own fitness.

The network initialization issue is also undone by Barabási and Albert [3]. Given realistic consideration, the individuals of an initial network are often small group but highly gathered and connected to each other. For instance, Advanced Research Projects Agency Network, the precursor of Internet, has only four vertices dispersed in four universities and connected to each others. Consequently, the initialization phase of a complex network is high clustering and has short distance.

Based on these issues, we then show a novel model based on the Poisson process that follows a power-law distribution.

A. Description of Proposed Model

The main idea of our proposed model is to propose the initial network, growth mechanism, preferential connection and end time, we summarize them as initialization, growth, connection, and termination and briefly introduce them as below.

1) *Initialization:* Given N vertices as a nearest-neighbor coupled networks, in which each vertex connects to its $K/2$ neighbors on the left and the right. For every pair of vertices, an extra connection is established by the probability of p.
2) *Growth:* Newly added vertices i are coming by the rate at λ. For each new vertex, η_i edges are connected to the existing vertices.
3) *Connection:* The probability of a newly added vertex connected to an existing vertex is decided by Π_i.
4) *Termination:* As the time reaches T, the algorithm terminates and the network is output.

In Initialization, an initial network is input as the size of m_0, and constructed as a NW small-world network, the values of K and p determine the degree of clustering, which can be controlled to simulate different types of networks. This step can solve the unaddressed issue of BA network.

The growth is the key section of our model, which is the main improvement of BA network, and the counting time starts from this phase to simplify calculation. The generation of vertices follows the rule of a Poisson distribution, and the detailed definition is given below.

Definition 1: The generation of new vertices $N(t), t \gg 0$ is a Poisson process having rate λ, $\lambda > 0$, and:
1) $N(0) = 0$;
2) the process has independent increments;
3) the number of added vertices in any interval of length t is Poisson distributed with mean λt, which is, for all s, $t \geq 0$

$$P\{N(t+s) - N(s) = k\} = \frac{[\lambda(t-s)]^k}{k!} e^{-\lambda(t-s)} \quad k = 0, 1, \ldots \tag{1}$$

Then, different from BA network, for each new vertex, the number of connections is supposed to follow a Gaussian distribution as its fitness function. The definition is expressed as follows.

Definition 2: For a newly added vertex i, the number of connections η_i is a Gaussian distribution with parameters μ and σ^2, and the density is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} 0 < x < +\infty \tag{2}$$

where x is supposed to be an integer.

The fitness function denotes the adaptivity of a new vertex, and the higher value indicates the vertex is easier to fit in with the network.

In connection and termination, Π_k, the probability of a connection to a vertex i, is denoted as

$$\Pi(k_i) = \frac{k_i}{\sum_j k_j} \tag{3}$$

where k_i is the degree of vertex i, i denotes all existing vertices. We utilize the roulette to select the vertex to be connected. Repeated connection is avoided. T is the time of termination, which can be controlled.

TABLE I
PARAMETERS FOR THE DEMONSTRATION

Parameter	N	K	p	λ	η	T
Value	5	2	0.1	1	1, 2	10

where N denotes the number of the initial network, K denotes the number of connections to neighbors, p is the probability of a new connection, λ denotes the rate of the generation, η is the number of connections, and T denotes the terminal time of generated vertex.

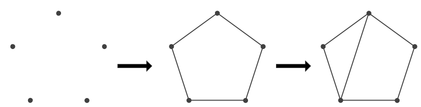


Fig. 1. Initialization: five vertices are connected as a nearest-neighbor coupled network, with each vertex connecting to its neighbor on the left and the right, and adding edges by the probability 0.1.

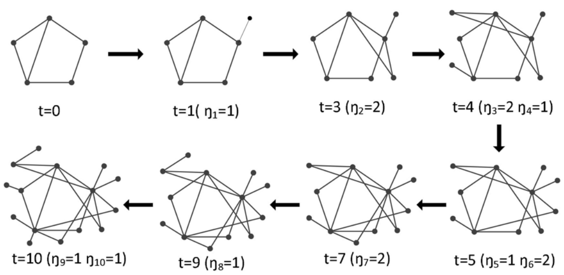


Fig. 2. HPN evolution. New vertices are generated by the mean rate 1 and connect to one or two existing vertex, the generation is terminated at time 10.

B. Demonstration of the Network Evolution

In this section, we present an example of the scale-free model based on a Poisson process and explain its evolution in details. In order to describe with clarity, the parameters are listed in Table I.

The detailed construction is illustrated in Figs. 1 and 2. Fig. 1 simulates the process of building an initial network, which is a simple small-world network and only one extra edge is connected as a shortcut. Fig. 2 contains the network evolution of the growth, connection and termination. From this figure, we can see that the mean rate $\lambda = (1 + 1 + 2 + 2 + 1 + 1 + 2)/10 = 1$. According to Definition 1, the generation follows a Poisson process with mean rate 1. And once a new vertex arrives, the number of connections to existing vertices is decided by a Gaussian distribution with the value $\eta_i \in \{1, 2\}$. Obviously, the poor vertices with the degree below 4 make the majority (80%) of this network, and the rich vertices with the degree upon 4 are in minority (20%), which is commonly referred as "the Matthew effect" or "the rich-get-richer phenomenon," and also the property of a power-law distribution.

C. Fitting Proposed Model in Network of Coauthorships

One of the potential applications of the HPN is to simulate the practical networks since they have many common characters such as the growth following the Poisson process, the connection following the Gauss distribution and the preferential attachment. In this section, we give a simple example to fit a HPN in the network of coauthorships.

The database is a network of coauthorships between scientists, and scientists in this case who are themselves publishing on the topic of networks, which was studied by Newman [21].

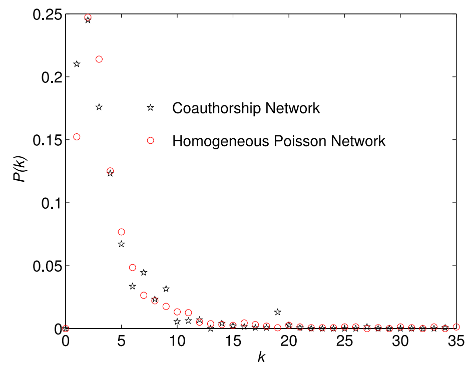


Fig. 3. Degree distribution of the coauthorship network and HPN, the values of $P(k)$ are (0, 0.1523, 0.2473, 0.2140, 0.1252, 0.0768, 0.0485, 0.0264, 0.0220, 0.0176, 0.0132, 0.0126, 0.0050, 0.0038, 0.0031, 0.0025, 0.0044, 0.0031, 0.0019, 0.0006, 0.0025, 0.0013, 0.0006, 0.0006, 0.0006, 0.0013, 0.0013, 0, 0.0006, 0, 0.0013, 0.0006, 0, 0.0013, 0) and (0, 0.1523, 0.2473, 0.2140, 0.1252, 0.0768, 0.0485, 0.0264, 0.0220, 0.0176, 0.0132, 0.0126, 0.0050, 0.0038, 0.0031, 0.0025, 0.0044, 0.0031, 0.0019, 0.0006, 0.0025, 0.0013, 0.0006, 0.0006, 0.0006, 0.0013, 0.0013, 0, 0.0006, 0, 0.0013, 0.0006, 0, 0.0013, 0).

To fit the real-world network, the parameters of growth and connection are required, and our object is to extract these data from the coauthorship network. As described in [21], the network started from two articles by Newman and Boccaletti, which indicate the initial network has only two connected vertices. The articles are published from 2003 to 2006 and a total of 1589 scientists are included in the network, then we suppose the growth rate $\lambda = 44.14$ per month and the termination time $T = 36$. From observation, we find the connection of each vertex is between 1 and 5, which can be considered as a Gauss distribution with $\mu = 3$ and $\sigma = 1$. With these arguments from the coauthorship network, we can build a HPN by our method.

After the theoretical model is obtained, we mainly compare its degree distribution with practical coauthorship network, the result is shown in Fig. 3. Then, we employ the correlation coefficient of their degree distributions to calculate the fitness of these two networks, that is

$$\rho_{d_c d_h} = \frac{\mathrm{cov}(d_c, d_h)}{\sqrt{\mathrm{Var}(d_c)}\sqrt{\mathrm{Var}(d_h)}} = 0.9772 \qquad (4)$$

where cov denotes the covariance, Var denotes the variance, d_c is the vector of the frequency ranging from degree 1 to 35 of the coauthorship network, and d_h is the vector of the frequency ranging from degree 1 to 35 of the homogeneous network, which are listed in the illustration of Fig. 3. The value of $\rho_{d_c d_h}$ indicates the fitness of vector d_c and d_h.

The result of correlation coefficient reveals that our proposed model has a good performance on fitting in the coauthorship network in the topology like the degree distribution. However, it is complex to decide whether an academic model is perfectly fitted in a practical network, and many factors should be taken into consideration and further study is required.

D. Theoretical Analysis of the Proposed Model

As we know, the scale-free is one of the most important topologies of complex networks. In this section, we focus on proving that the proposed model is a scale-free network, which is, given a fixed λ, the degree exponent of this network is unrelated to the construction time. We utilize two approaches to accomplish the proof. Before the proof carries out, some important definitions are provided first.

Definition 3: At time t, the probability that the degree of a random vertex in the network is exactly k, denoted by $P_k(t)$, is said to be a transient degree distribution.

Definition 4: Assuming that $P_k(i,t)$ denotes the probability that, at time t, the degree of vertex i is exactly k, and the initial network is ignored, then the average probability

$$P_k(t) = \frac{\sum_i P_k(i,t)}{t} \tag{5}$$

is called the average degree distribution for the network at time t.

Definition 5: If $\lim_{t\to\infty} P_k(t)$ exists and is equal to $P(k)$, then $P(k)$ is said to be the steady state degree distribution of this network.

Definitions 3 and 4 provide two different forms of the degree distribution at time t, and Definition 5 is the target of our proof. As $t \to \infty$, the small initial network has little effect on the degree distribution, therefore, in the proof below, the initial network is ignored, which is, $t = 0$, the number of vertex is 0. All the lemma and theorems proven below are based on the homogeneous Poisson model.

Lemma 1: Given a fixed arrival rate λ and an expected connection μ, the expectation of total degree at time t is $2\lambda\mu t$.

Proof: Suppose that $N(t)$ denotes the number of vertices at time t, η denotes the number of connections, then $E[N(t)\eta]$ denotes the total number of edges at time t, and by definition, $E[N(t)] = \lambda t$ and $E[\eta] = \mu$.

The number of vertices is independent of connections, therefore, the covariance is 0, that is

$$\operatorname{cov}(N(t),\eta) = E[N(t)\eta] - E[N(t)]E[\eta] = 0 \tag{6}$$

scilicet

$$E[N(t)\eta_i] = E[N(t)]E[\eta_i] = \lambda\mu t. \tag{7}$$

The total degree is two times of the total edges, and according to the properties of expectation

$$E = E[2N(t)]E[\eta_i] = 2\lambda\mu t. \tag{8}$$

The result follows. ■

Theorem 1: Given a fixed arrival rate λ, the degree exponent of the proposed network is independent of time.

The first proof derives from the dynamics method which is widely used by physicists.

Proof: Suppose that $k_i(t)$ is the degree of vertex i at time t. According to Lemma 1, at time t, the total degree is

$$\sum_j k_j \approx 2\,\lambda\mu t. \tag{9}$$

For each new vertex with η connections, the probability that an edge connects to the existing vertex i is approximated as

$$P = \binom{\eta}{1}[\Pi(k_i)][1-\Pi(k_i)]^{\eta-1} \approx \eta\Pi(k_i) = \frac{\eta k_i}{2\lambda\mu t} \tag{10}$$

where $\Pi(k_i)$ is defined by (3).

As we know, λ new vertices generate per unit time, and the degree growth rate of vertex i is λP. Then, $k_i(t)$ can be regarded as a dynamic equation based on the continuum theory, and follows:

$$\begin{cases} \dfrac{\partial k_i(t)}{\partial t} \approx \lambda P = \dfrac{\eta k_i(t)}{2\mu t} \\ k_i\left(\dfrac{i}{\lambda}\right) = \eta \end{cases} \tag{11}$$

the solution is

$$k_i(t) = \eta\left(\frac{\lambda t}{i}\right)^{\frac{\eta}{2\mu}} \tag{12}$$

where $\eta/2\mu$ is the kinetic index whose expectation is 1/2.

Assuming that vertex i is randomly selected from λt vertices and follows a homogeneous distribution, that is, $p(i) = 1/\lambda t$, according to (12), the transient degree distribution is:

$$P\{k_i(t) < k\} = P\left\{ i > \lambda t\left(\frac{k}{\eta}\right)^{-\frac{2\mu}{\eta}} \right\} = 1 - \frac{\lambda t\left(\frac{k}{\eta}\right)^{-\frac{2\mu}{\eta}}}{\lambda t} \tag{13}$$

the partial derivatives by k is

$$P_k(t) = \frac{\partial P\{k_i(t) < k\}}{\partial k} = 2\mu\eta^{\frac{2\mu}{\eta}-1}k^{-\left(\frac{2\mu}{\eta}+1\right)} \tag{14}$$

then, the steady state distribution is

$$P(k) = \lim_{t\to\infty} P_k(t) = 2\mu\eta^{\frac{2\mu}{\eta}-1}k^{-\gamma} \tag{15}$$

where $\gamma = (2\mu/\eta) + 1$ and is independent of time t.

The result follows. ■

The alternative proof is based on the probabilistic approach and commonly used by mathematicians.

Proof: Suppose that $k_i(t)$, the degree of vertex i at time t, denotes a stochastic variable, and $P_k(i,t)$ denotes the probability that the degree of vertex i is k at time t. Suppose that per unit time, λ new vertices are generated, each with η connections, then the probability that an edge connects to the existing vertex i is approximated as

$$P' = \binom{\lambda}{1}P(1-P)^{\lambda-1} \approx \lambda P \approx \frac{\eta k}{2\mu t} \tag{16}$$

where P follows (10).

Hence, the connectivity distribution of a vertex obeys the following master equation:

$$P_k(i,t+1) \approx \frac{\eta(k-1)}{2\mu t}P_{k-1}(i,t) + \left(1 - \frac{\eta k}{2\mu t}\right)P_k(i,t) \tag{17}$$

that is

$$\begin{aligned} &2\mu t[P_k(i,t+1) - P_{k-1}(i,t)] \\ &\qquad = -\eta[kP_k(i,t) - (k-1)P_{k-1}(i,t)]. \end{aligned} \tag{18}$$

Equation (18) is a difference equation, and based on the continuum theory, can be regarded as a partial differential equation, which is

$$2\mu t\frac{\partial P_k(i,t)}{\partial t} = -\eta\frac{\partial[kP_k(i,t)]}{\partial k}. \tag{19}$$

Assuming that the expected degree function of vertex i is

$$\bar{k}(i,t) = \sum_{k=0}^{\infty} kP_k(i,t) \approx \int_0^{\infty} kP_k(i,t)dk. \tag{20}$$

Then, by integral definition of $\bar{k}(i,t)$ and the integration by parts of definite integral

$$\begin{aligned}\bar{k}(i,t) &= -\left\{0 - \int_0^{\infty} kP_k(i,t)dk\right\}\\ &= -\left\{\left[k^2P_k(i,t)\right]_0^{\infty} - \int_0^{\infty} kP_k(i,t)dk\right\}\\ &= -\int_0^{\infty} kd[kP_k(i,t)] = -\int_0^{\infty} kdk\frac{\partial[kP_k(i,t)]}{\partial k}.\end{aligned} \tag{21}$$

Equation (19) can be converted into

$$2\mu t\int_0^{\infty} kdk\frac{\partial P_k(i,t)}{\partial t} = -\eta\int_0^{\infty} kdk\frac{\partial[kP_k(i,t)]}{\partial k} \tag{22}$$

considering (21), that is

$$\frac{\partial\bar{k}(i,t)}{\partial t} = \frac{\eta\bar{k}(i,t)}{2\mu t}. \tag{23}$$

Thereupon, the partial differential equation turns into

$$\begin{cases}\dfrac{\partial\bar{k}(i,t)}{\partial t} = \dfrac{\eta\bar{k}(i,t)}{2\mu t}\\ \bar{k}\left(i,\dfrac{i}{\lambda}\right) = \mu\end{cases} \tag{24}$$

its general solution is $\bar{k} = t^{\eta/2\mu}C$, and with the substitution of the condition, the result is

$$\bar{k} = \mu\left(\frac{\lambda t}{i}\right)^{\frac{\eta}{2\mu}} \tag{25}$$

and we can obtain

$$i = \lambda t\left(\frac{\mu}{k}\right)^{\frac{2\mu}{\eta}}. \tag{26}$$

The discrete distribution is described as a continuous Dirac δ-function, then according to Definition 4, our degree distribution can be denoted as

$$\begin{aligned}P_k(t) &= \frac{1}{t}\sum_i^{\lambda t} P_k(i,t) = \frac{1}{t}\int_0^{\lambda t} P_k(i,t)\\ &= \frac{1}{t}\int_0^{\lambda t}\delta\big(k-\bar{k}_i(t)\big)di = -\frac{1}{t}\int_0^{\lambda t}\delta\big(\bar{k}_i(t)-k\big)di\end{aligned} \tag{27}$$

obviously

$$d\bar{k}_i(t) = -\frac{\partial\bar{k}_i(t)}{\partial i}di \tag{28}$$

substitute it into (27), we obtain that

$$\begin{aligned}P_k(t) &= -\frac{1}{t}\left(1\bigg/\frac{\partial\bar{k}_i(t)}{\partial i}\right)\int_0^{\lambda t}\delta\big(\bar{k}_i(t)-k\big)d\bar{k}_i(t)\\ &= -\frac{1}{t}\left(1\bigg/\frac{\partial\bar{k}_i(t)}{\partial i}\right) = \frac{2i}{\eta t}\left(\frac{i}{\lambda t}\right)^{\frac{\eta}{2\mu}}.\end{aligned} \tag{29}$$

According to Definition 5, (26), and (29), the steady state degree distribution is

$$\begin{aligned}P(k) &= \lim_{t\to\infty} P_k(t) = \lim_{t\to\infty}\left(\frac{2i}{\eta t}\left(\frac{i}{\lambda t}\right)^{\frac{\eta}{2\mu}}\right)\Bigg|_{i=\lambda t\left(\frac{\mu}{k}\right)^{\frac{2\mu}{\eta}}}\\ &= 2\lambda\mu^{1+\frac{2\lambda\mu}{\eta}}\eta^{-1}k^{-\gamma}\end{aligned} \tag{30}$$

where $\gamma = (2\mu/\eta) + 1$ and is independent of time t.

The result follows. ■

Due to the different proof approaches, (15) and (30) display a little difference, but if we seek their expectations, the result is the same

$$P(k) = 2\lambda\mu^{2\lambda}k^{-\bar{\gamma}} \tag{31}$$

where $\bar{\gamma} = 3$.

Obviously, when λ is fixed, the conclusion is that our proposed model is independent of time and follows the conception of a scale-free network.

However, this is just a fundamental and ideal model, many variables should be taken into consideration, such as variable rates and mortality rate of vertices. Therefore, based on the homogeneous Poisson scale-free model, we propose two extension models and discuss whether they are still free of time.

III. Various Extensions of the Poisson Scale-Free Model

As we discuss above, the generation rate of many complex networks changes all the time. In different periods, the rate is also distinct, for example, the population rate during war years is quite different from the peace time. This phenomenon of complex networks is a typical nonhomogeneous process. Additionally, most of complex models are simply pure birth processes which means the vertices are always increasing, but in fact, individual not only generates but vanishes in any kind of networks, for instances, the species extinction in food chain networks, the bankruptcy of a company in banking network, all of these networks present the birth and death characteristics.

Based on these two viewpoints, in this section, we propose two various extension model: 1) the NHPN model and 2) the BDN model.

A. Nonhomogeneous Poisson Network

The construction algorithm of this model is close to Section II-A: initialization, growth, connection, and termination, the only difference is in growth.

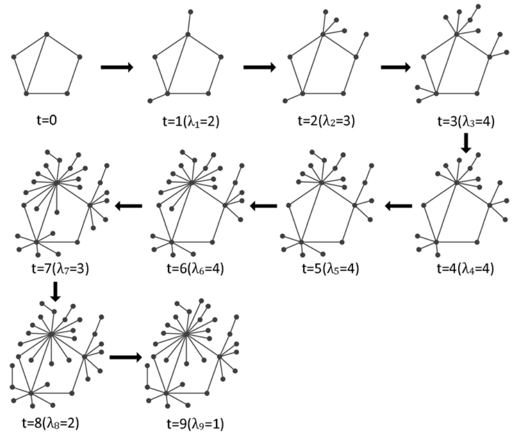


Fig. 4. NHPN evolution. New vertices are generated by the rate $\lambda(t)$ and connect to one existing vertex, the generation is terminated at time 9.

1) Growth: Newly added vertices i are coming by the rate at $\lambda(t)$. For each new vertex, η_i edges are connected to the existing vertices.

This process with $\lambda(t)$ which relates to the time t can be regarded as a nonhomogeneous Poisson process, the definition is as follows.

Definition 6: The generation of new vertices $N(t), t \gg 0$ is a nonhomogeneous Poisson process with the continuous function $\lambda(t)$, $t \geq 0$, and:

1) $N(0) = 0$;
2) the process has independent increments;
3) $P\{N(t + \Delta t) - N(t) = 1\} = \lambda \Delta t + o(t)$;
4) $P\{N(t + \Delta t) - N(t) \geq 2\} = o(t)$.

Theorem 2: If $\{N(t), t \geq 0\}$ is a nonhomogeneous Poisson process and $\lambda(t)$ is continuous, in interval $[t, t + s]$, the probability that the number of added vertices is k follows:

$$P\{N(t+s) - N(s) = k\} = \frac{[m(t+s) - m(t)]^k}{k!} e^{m(t+s)-m(t)} \quad k = 0, 1, \ldots \tag{32}$$

where $m(t) = \int_0^t \lambda(s) ds$, and the $m(t + s) - m(t)$ is the mean value of this process.

Here, we display a model constructing process, the continuous $\lambda(t)$ is expressed as follows:

$$\lambda(t) = \begin{cases} t + 1, & 0 < t \leq 3 \\ 4, & 3 < t \leq 6 \\ 6 - (t - 4), & 6 < t \leq 9 \end{cases}$$
$$\lambda(t) = \lambda(t - 9). \tag{33}$$

The initial network refers to Fig. 1. For simplicity, we assume that $\eta_i = 1$ and the shortest unit time is 1. The process is demonstrated in Fig. 4. The power-law property is displayed in this network, low degree vertices below 4 are in the majority (about 91.67%), while high degree vertices above 7 are in the minority (about 5.56%).

Next, we focus on the degree distribution of the NHPN. Suppose that $\lambda(t)$ is denoted as (33), and for clarity, we assume that number of edges added each time is a fixed μ and ignore the initial network. The dynamic method is applied to solve this issue.

Suppose that $k_i(t)$ is the degree of vertex i at time t, the total number of added vertices is $m(t)$, then the total degree is

$$\sum_j k_j \approx 2\, m(t)\mu. \tag{34}$$

The probability that an edge connects to the existing vertex i is approximated as

$$P = C_\mu^1 [\Pi(k_i)][1 - \Pi(k_i)]^{\mu-1} \approx \mu \Pi(k_i) = \frac{k_i}{2m(t)}. \tag{35}$$

The degree growth rate of i per unit time is $\lambda(t)P$, and dynamic equation based on the continuum theory is expressed as

$$\begin{cases} \dfrac{\partial k_i(t)}{\partial t} \approx \dfrac{k_i \lambda(t)}{2m(t)} \\ k_i\Big(m^{-1}(i)\Big) = \mu \end{cases} \tag{36}$$

where $m^{-1}(i)$ denotes the time vertex i arrives. Notice that $(1/k)\partial k = (\partial m(t)/2m(t))$, the general solution is $k_i(t) = m(t)^{(1/2)} C$, and with the substitution of the condition, the result is

$$k_i(t) = \mu \left[\frac{m(t)}{i}\right]^{\frac{1}{2}}. \tag{37}$$

Assuming that vertex i is randomly selected from $m(t)$ vertices and follows a homogeneous distribution, that is, $p(i) = 1/m(t)$, the transient degree distribution is:

$$P\{k_i(t) < k\} = P\left\{i > m(t)\left(\frac{\mu}{k}\right)^2\right\} = 1 - \frac{m(t)\left(\frac{\mu}{k}\right)^2}{m(t)} \tag{38}$$

the partial derivatives by k is

$$P_k(t) = \frac{\partial P\{k_i(t) < k\}}{\partial k} = 2\mu^2 k^{-3} \tag{39}$$

then, the steady degree distribution is

$$P(k) = \lim_{t \to \infty} P_k(t) = 2\mu^2 k^{-\gamma} \tag{40}$$

where $\gamma = 3$ and is independent of time.

From this derivation, we can see that no matter homogeneous or nonhomogeneous the generation rate is, as long as it is continuous, the degree distribution keeps stable, and time range has no influence on the distribution.

B. Birth and Death Network

The last network model is more complicated than previous models, which takes both the generation of vertices and their extinctions into consideration, and we call it BDN model.

So far, most of network models for complex systems ignore the degenerations of vertices, but these phenomena really exist and have a significant influence on the network topology. Therefore, we should ponder both generation and degeneration. In this model, the key issue is the death process. And in the view of practical situation, the death follows the rule that "the rich live while the poor die," or "the rich are getting richer, and the poor poorer." That is to say, Barabási and Albert [3] only consider half of the Matthew effect [31], and we extend it to the aspect of poor.

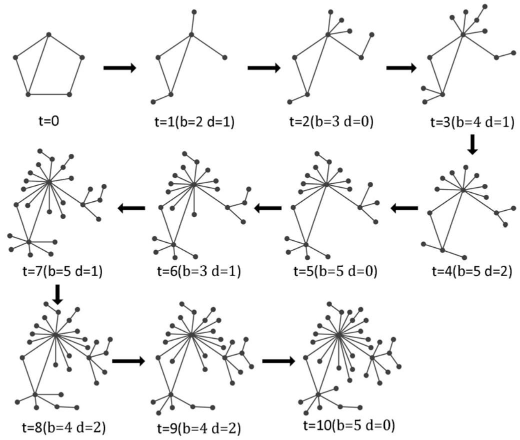


Fig. 5. BDN evolution. New vertices are generated by the rate λ and connect to one existing vertex, old vertices degenerate by the rate ν, the process is terminated at time 10.

The death mechanism is the crucial issue for BDNs. We assume that the death time follows an exponential distribution with parameter μ, indicating that the degeneration of old vertices is also a Poisson process. The selection of one vertex to die follows the Matthew effect, that is, the lower degree vertex is more likely to die. To solve Γ, the probability of existing vertices to die, the value convert $\widetilde{k_i}$ is required, that is

$$\widetilde{k_i} = \frac{k_{\max} - k_i}{k_{\max} - k_{\min}} \tag{41}$$

$\widetilde{k_i}$ is also said to be the poor degree. Obviously, the low degree vertex has a high value of $\widetilde{k_i}$.

Then, Γ_i is denoted as

$$\Gamma_i = \frac{\widetilde{k_i}}{\sum_j \widetilde{k_j}} \tag{42}$$

which ensures that the poor is more likely to be perished and $\sum_i \Gamma_i = 1$. Then, we utilize roulette to select the vertex to be perished. By this mechanism, the degeneration process is confirmed.

The complete construction process is as follows.

1) *Initialization:* Given N vertices as a nearest-neighbor coupled networks, in which each vertex connects to its $K/2$ neighbors on the left and the right. For every pair of vertices, an extra connection is established by the probability of p.
2) *Growth and Extinction:* Newly added vertices i are coming by the rate at λ. For each new vertex, η_i edges are connected to the existing vertices. The existing vertices are gone by the rate at ν.
3) *Connection and Disconnection:* The probability of a newly added vertex connected to an existing vertex is decided by Π_k. The probability of an existing vertices to die is decided by $\Gamma(k_i)$.
4) *Termination:* Once the time reaches T, the algorithm terminates and the network is output.

We demonstrates a BDN evolution. The initial network refers to Fig. 1, and each new vertex connects to one existing vertex. The process is displayed in Fig. 5.

Unfortunately, the dynamic method and probabilistic approach cannot obtain the solution of the degree distribution of BDN, we still try to work it out and require further studies.

IV. Simulation and Analysis

In this section, we focus on modeling process and displaying the statistical degree distribution of proposed network models, besides, the robustness and vulnerability simulations are carried out.

A. Network Modeling

Modeling of network is the foundation of our simulation, and the random process is the linchpin of modeling. Therefore, we mainly describe the simulation of random processes in this section.

The key of an HPN is to simulate a Poisson process with λ, in other words, a mutually independent exponential distributed random sequence with expectation $1/\lambda$ is required. Then, our method is as follows.

1) Set the values of λ and $T_{\max}$.
2) Generate exponential distribution random values with λ, denoted as t_i.
3) If cumulative time $T_i \leq T_{\max}$, let $T_i = T_i + t_i$, else output the temporal series.

The temporal series of an NHPN is produced by the rarefaction method, that is, assuming $\lambda(t) \leq \lambda$, where λ is a constant, $\{t_1, t_2, \ldots\}$ is the temporal series of a homogeneous Poisson process with λ, for each t_i, with the probability $(\lambda(t_i)/\lambda)$ to save and $1 - (\lambda(t_i)/\lambda)$ to abandon, then, the saved temporal series $\{t'_1, t'_2, \ldots\}$ follows the nonhomogeneous Poisson with $\lambda(t)$. The detailed process is as follows.

1) Produce the temporal series with parameter λ, denoted as $\{t_1, t_2, \ldots, t_n\}$.
2) Generate a random number r_i, if $r_i \leq (\lambda(t_i)/\lambda)$, t_i is saved, else t_i is abandoned.
3) For those saved t_i, remark them as $\{t'_1, t'_2, \ldots, t'_k\}$ and output.

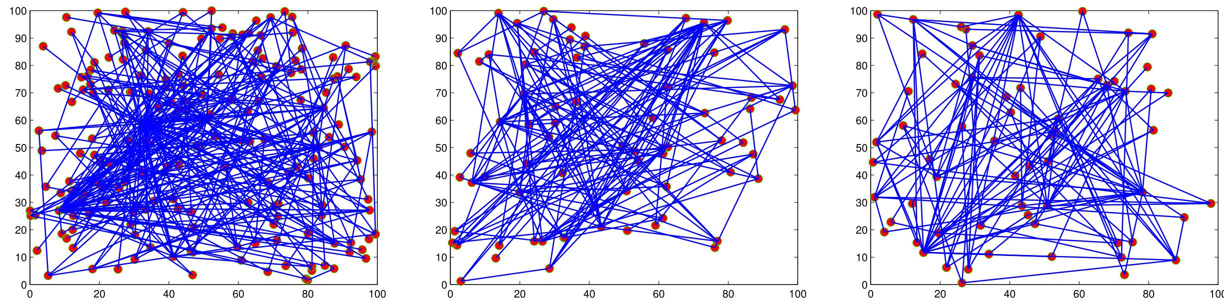
For BDN, we employ a queuing system $M/M/1/\infty/\infty$. The temporal series of generation and degeneration both follows exponential distribution random sequences with expectation $1/\lambda$ and $1/\nu$. The system decides to serve the vertex by the rule "poorer die" with rate ν, and records the leaving time. The detailed process is as follows.

1) Produce the temporal series with parameter λ, denoted as $\{t_1, t_2, \ldots, t_n\}$.
2) The system serves the vertices by the rate ν, records their leaving time, $\{T_1, T_2, \ldots, T_k\}$.
3) When $T_{\max}$ is reached, output the temporal series.

Based on these counting processes, we display three network models shown in Fig. 6.

B. Degree Distribution

Degree is a simple but significant character for a single vertex in network, and the degree distribution is the most important property and topology for complex networks.



(a) (b) (c)

Fig. 6. Illustration for three different networks. Initial networks all start with ten nodes, each node connects to its two neighbors on the left and right, with the probability 0.2 to add a path to the other node. The settings are (a) $\lambda = 4$, (b) $\lambda(t) \le \lambda = 4$, and (c) $\lambda = 4$ and $\mu = 1$. The termination time $T_{\max} = 50$.

In this section, we use the statistical method to analyze degree distribution and obtain a degree exponent from empirical data.

As we know, the degree distribution of most scale-free networks including our proposed models obeys the power-law distribution, which can be mathematically denoted as

$$y = \beta x^{-\gamma} \tag{43}$$

where β is a normalization, and γ the power-law exponent.

Discrete empirical data, as circles in Fig. 7, are only approximations of continuous mathematical power-law in line. The task for us is to employ a simple power-law to more precisely describe the data, thus we need to remove those complex factors as noisy data. There are many proposed algorithms to solve this issue, such as the logarithmic binning [22], maximum likelihood estimation [23], [24], and so on [25]. Here, we apply a numerical method.

From Fig. 7, we can see that, for the large degree k, the deviation from the mathematical value is obvious. These tail values are often caused by the statistical nature of the empirical distribution, which appears that, at large k, the number of vertices are one or zero, which causes noticeable noises. To get rid of them, we set a noise threshold N for the data, we remove Nt values from the empirical data as the statistical noise. For example, for a data set with 1000 points, if $N = 0.005$, that means we need to remove five highest degree from this data as noises, see tail noises in red in Fig. 7. The value of N varies in different networks to remove the long tail, generally, 0.5%–2% in this paper. In addition, since the connection follows a Gauss distribution $\eta \sim N(\mu, \sigma^2)$, the value of k below $\mu + 3\sigma$ also leads to noticeably noisy appearance, therefore, these points are also removed, see head noises in black in Fig. 7.

After noises are removed, the line fitting is required. Assuming that the empirical data follow (43), as we operate in log coordinate, the equation is logged as:

$$A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = b \tag{44}$$

where $A = (\log x, 1)$, $x_1 = -\gamma$, $x_2 = \log \beta$, and $b = \log y$.

To get the best fitting line, we need to seek for the least squared error, equationally, that is

$$\begin{cases} \min \|b - A\bar{x}\|_2 \\ \text{s.t. } A^T A\bar{x} = A^T b \end{cases} \tag{45}$$

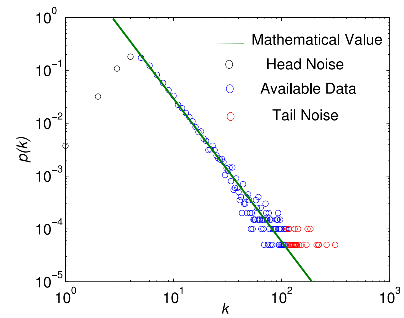


Fig. 7. Sample of deviations from a mathematical power-law. The continuous line is a mathematical power-law with $\gamma = 3$, and discrete points denote the empirical data.

where $\bar{x} = (x_1, x_2)$. x_1 in solution $\bar{x}$ for (45) is the slope of the best fitting line. With this argument, we can determine the plot of the degree distribution of a network.

By utilizing this method, we plot degree distributions of three network models in double logarithmic coordinates and obtain the slope, which is also the degree exponent. In this simulation, we mainly consider three variables: the input rate, connection and time. Specifically, for HPN, λ, μ, and t are measured; for NHPN, $\lambda(t) \le \lambda$, μ and t are measured; for BDN, λ, $\nu_d = 1/3$, μ, and t are measured. One argument is adjusted while other two are constant, so that we can observe whether the degree exponent has a relationship with these arguments. Detail argument settings and results of degree exponents are listed in Table II including HPN, NHPN, and BDN. For each, we increase λ from 4 to 12, μ from 10 to 50, and t from 1000 to 5000, the corresponding degree exponents γ are also listed.

From this table, we can draw the conclusion that, for all three networks, the value range of degree exponents lies in $[2.7, 3.0]$, which is independent of λ, μ, and t.

We also demonstrate the degree distributions and exponents in double logarithmic in Figs. 8–10. For clarity, we only choose three values for each situation. Three figures show the same trend, even with different λ [Figs. 8(a)–10(a)], μ [Figs. 8(b)–10(b)], and t [Figs. 8(c)–10(c)], indicating that they are all typical scale-free networks. However, due to the rarefaction method we use, with the same argument, the nonhomogeneous network possesses less vertices, therefore, Fig. 8 shows fewer points than other two figures. And for

TABLE II
DEGREE EXPONENTS FOR THREE KINDS OF NETWORKS WITH DIFFERENT ARGUMENTS

HPN						NHPN						BDN					
$\mu = 10$ $t = 1000$		$\lambda = 4$ $t = 1000$		$\lambda = 4$ $\mu = 10$		$\mu = 10$ $t = 1000$		$\lambda = 4$ $t = 1000$		$\lambda = 4$ $\mu = 10$		$\mu = 10$ $t = 1000$		$\lambda = 4$ $t = 1000$		$\lambda = 4$ $\mu = 10$	
λ	γ	μ	γ	t	γ	λ	γ	μ	γ	t	γ	λ	γ	μ	γ	t	γ
4	2.7	10	2.9	1000	2.9	4	2.8	10	3.0	1000	2.9	4	3.0	10	2.9	1000	2.9
6	2.8	20	2.8	2000	2.8	6	3.0	20	2.8	2000	2.7	6	2.9	20	2.9	2000	2.8
8	2.8	30	2.8	3000	2.9	8	2.9	30	3.0	3000	2.8	8	3.0	30	2.8	3000	2.8
10	2.9	40	2.7	4000	2.9	10	2.8	40	2.8	4000	2.8	10	3.0	40	3.0	4000	2.8
12	2.9	50	2.8	5000	3.0	12	2.8	50	2.9	5000	3.0	12	2.8	50	3.1	5000	2.8

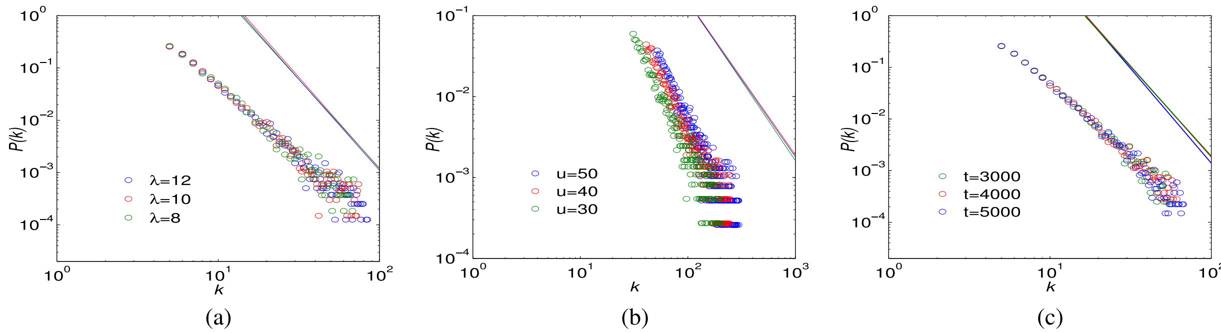


Fig. 8. Degree distributions of HPNs with different arguments in logarithmic coordinates. Arguments are set as (a) $\mu == 10$ and $t == 1000$, (b) $\lambda == 4$ and $t = 1000$, and (c) $\lambda == 4$ and $\mu == 10$.

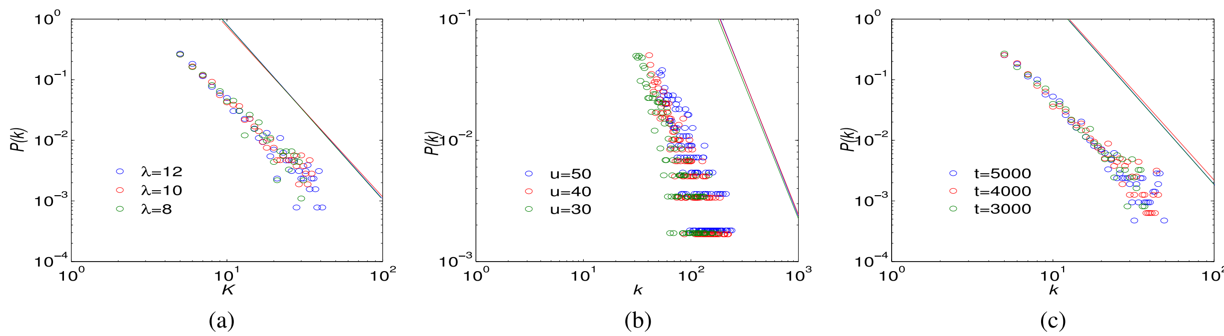


Fig. 9. Degree distributions of NHPNs with different arguments in logarithmic coordinates. Arguments are set as (a) $\mu == 10$ and $t == 1000$, (b) $\lambda == 4$ and $t = 1000$, and (c) $\lambda == 4$ and $\mu == 10$.

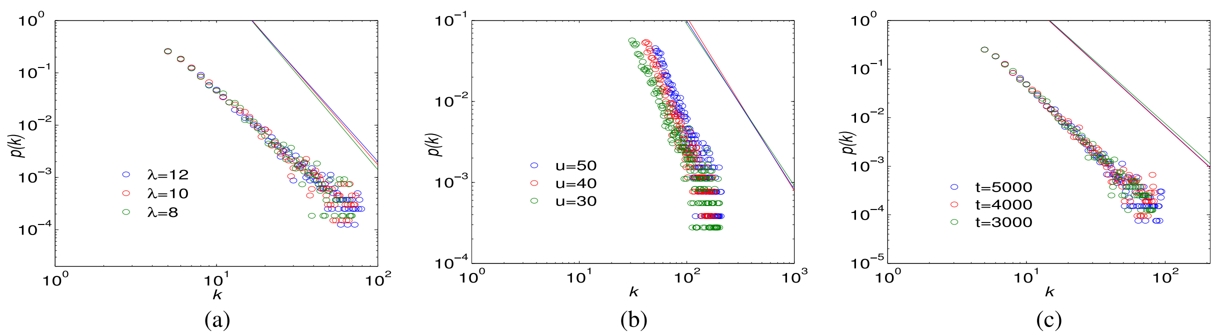


Fig. 10. Degree distributions of BDNs with different arguments in logarithmic coordinates. Arguments are set as (a) $\mu == 10$ and $t == 1000$, (b) $\lambda == 4$ and $t = 1000$, and (c) $\lambda == 4$ and $\mu == 10$.

the same type of network, under the condition of same μ, different λ and t display the approximate result since both two arguments increase total number of vertices of networks [see Figs. 8(a) and (c)–10(a) and (c)]. In addition, under the condition of same λ and t, higher μ makes the points wider, the reason is that μ determines the number of connections, the high value will increase the degree of vertex, and enlarge the distance of different degrees k, which makes the plot fatter [see Figs. 8(b)–10(b)].

In brief, getting rid of the noise in the beginnings and tails of empirical distributions, all three types of networks display a good power-law character and follow our theoretical analysis.

C. Reliability of Networks

Complex network reliability indicates the ability of a complex network to perform its original function even damaged by natural or man-made attacks. Evaluations for reliability

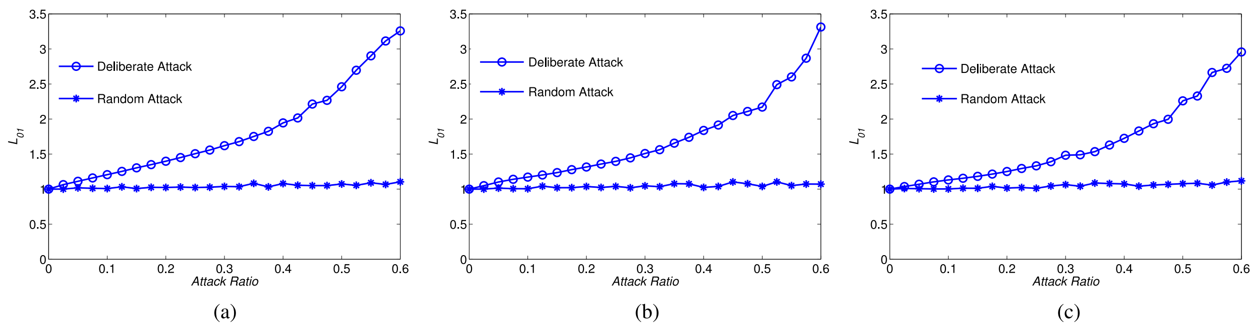


Fig. 11. Deliberate and random attack with different ratios to networks. Abscissa shows the ratio of attack and ordinate shows variation of average path length after the attack L_{01}. (a) HPN. (b) NHPN. (c) BDN.

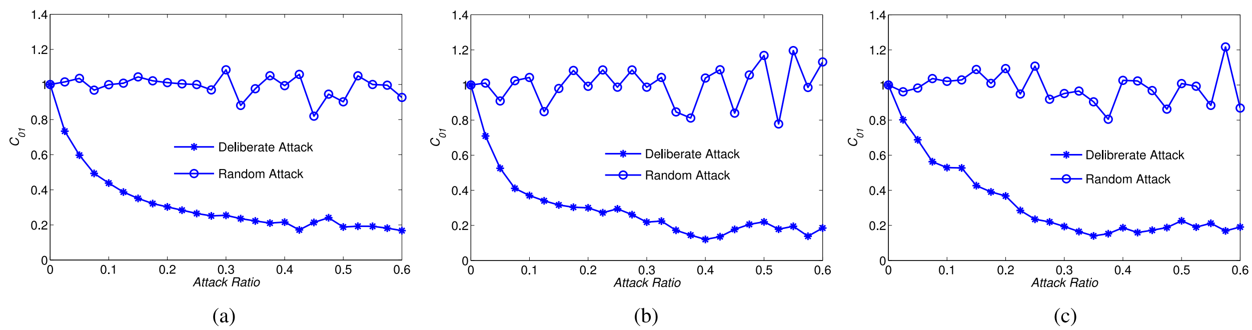


Fig. 12. Deliberate and random attack with different ratios to networks. Abscissa shows the ratio of attack and ordinate shows variation of average clustering coefficient after the attack C_{01}. (a) HPN. (b) NHPN. (c) BDN.

are invulnerability and survivability, generally, measured by average path length and clustering coefficient.

The first argument is the average path length, the reason for choosing it is that it can perform the connectivity of network. In general, the lower the average search step is, the better the connectivity. Therefore, comparing the network before and after the attack, we can learn the strength and hazard of the attack [26]. Suppose l_0 denotes the average path length of the original network, l_1 denotes the average path length after the first attack, then

$$L_{01} = \frac{l_1}{l_0} \tag{46}$$

where L_{01} indicates the strength and hazard of the first attack.

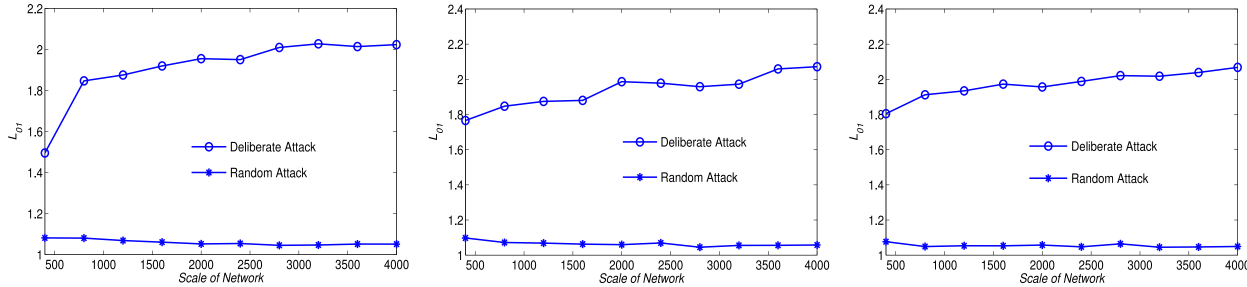
Analogously, another argument for measuring reliability is the clustering coefficient. As we know, the clustering coefficient of one vertex indicates its local connectivity, so the average clustering coefficient can denote the connectivity of the whole network. Suppose c_0 denotes the average clustering coefficient of the original network, c_1 denotes the average clustering coefficient after the first attack, then

$$C_{01} = \frac{c_1}{c_0} \tag{47}$$

where C_{01} denotes the strength and hazard of the first attack.

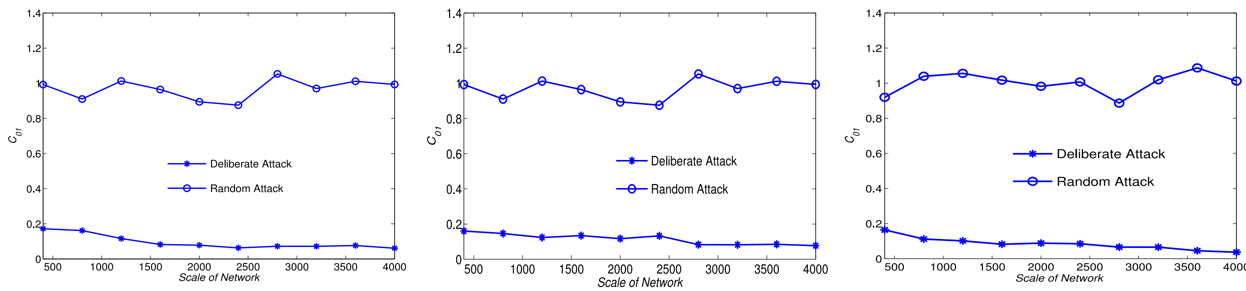
The other issue is the attack itself. Generally, there are two kinds of attacks: 1) deliberate attack and 2) random attack. The deliberate attack operates with certain strategies to remove vertices and edges, and is usually caused by human factors. Contrarily, the random attack removes vertices and edges with no purpose, and is always caused by self factors.

The first experiment runs in the invariable scale of three kinds of networks. The execution time of all networks is 500, For HPN, the input rate $\lambda = 4$, for NHPN, the input rate $\lambda(t) \leq \lambda = 4$ which follows (33), and for BDN, the input rate $\lambda = 4$ and output rate $\nu = 1$. For each network, we record its original average path length and clustering coefficient as l_0 and c_0, then, attack networks deliberately and randomly and record the length and coefficient after attack as l_1 and c_1. Equations (46) and (47) are employed to calculate the changing rate, the results of deliberate and random attack are displayed in Figs. 11 and 12. From these two figures, we can obviously see that all three networks display great vulnerability to deliberate attack. The change of average path length increases intensively with the growth of attack ratio while average clustering coefficient drops rapidly, the underlying reason is the inhomogeneity of degree distribution, so that the less vertices occupy the most degrees, once they were removed from network, most connections are cut off, shortcuts are also destroyed and gathering degree decreases, on the diagram, it displays the L_{01} grows [see Fig. 11(a)–(c)], those plots marked by circles, and C_{01} drops [see Fig. 12(a)–(c)], those plots marked by circles. Unlike the deliberate attack, all three networks display strong robustness to random attack, both the average path length and clustering coefficient keep steady, since the most vertices occupy the low degree, which is less important in the whole network, and the random attack mostly works on these low degree vertices. Therefore, the connectivity and clustering of the whole network hardly change, on the diagram, it displays that L_{01} and C_{01} keep as straight lines [see Figs. 11(a)–(c) and 12(a)–(c)], those plots marked by stars.



(a) (b) (c)

Fig. 13. Deliberate and random attack to networks with different scales. Abscissa shows the scale of network and ordinate shows variation of average path length after the attack L_{01}. (a) HPN. (b) NHPN. (c) BDN.



(a) (b) (c)

Fig. 14. Deliberate and random attack to networks with different scales. Abscissa shows the scale of network and ordinate shows variation of average clustering coefficient after the attack C_{01}. (a) HPN. (b) NHPN. (c) BDN.

The other experiment focuses on the relationship between reliability and scale of the network. The argument setting is as follows: for HPN, the input rate $\lambda = 4$, for NHPN, the input rate $\lambda(t) \leq \lambda = 4$ which follows (33), and for BDN, the input rate $\lambda = 4$ and output rate $\nu = 1$. The scales of network are set as 400, 800, 1200, 1600, 2000, 2400, 2800, 3200, 3600, and 4000. The attack ratio is 0.5 which means half of vertices are removed. Plots of L_{01} and C_{01} are displayed in Figs. 13 and 14. From these figures, we can see that the larger the networks are, the less stable they are. The reason is that, in the constructing processes of all three networks, the more vertices added to network means that the highest degree will increase continuously. Deliberate attack first aims at these highest degree vertices, once they are removed from the network, the distance of their neighbors will potentially increase [see Fig. 13(a)–(c)], those plots marked by circles, and the clustering will decrease [see Fig. 14(a)–(c)], those plots marked by circles. However, with the increasing of network scale, random attack has a very low probability to remove high degree vertex, so the large network shows robustness to it [see Figs. 13(a)–(c) and 14(a)–(c)], those plots marked by stars.

Both two simulations show that three proposed network models are "robust yet fragile," which is a basic and significant character for complex systems and scale-free networks [27], [28].

V. Conclusion

Network modeling and degree distribution, which are the most significant research domains for complex networks, demand greater attention and further study. In this paper, we introduce three novel network models, based on the homogeneous Poisson, nonhomogeneous Poisson, and birth and death process. Additionally, we analyze their degree distribution, and obtain degree exponents by applying the dynamics method and probabilistic approach. Conclusively, simulations of network construction, statistical degree distribution and network reliability are carried out to show that our models are typical scale-free networks.

This paper reveals that, regardless of the input rate and connections of individuals, once the network constructing mechanism is decided, the topology such as the degree distribution of complex networks remains unchanged. More practically, the war or financial crisis will not change the scale-free character of social or financial networks, which can be described as a nonhomogeneous Poisson process for complex systems applied to most real networks. Further, the λ and t determine the scale of network, and our proof indicates that the degree exponent is free of these two arguments, that is to say, no matter the larger scale or smaller scale (which may be considered as the subgraph of the large) networks are, their degree distributions are identical. In that sense, we can study the small or original type of complex systems to reveal the topology of larger scale. Additionally, our models with different input rates follow and further confirm the definition of complex systems and networks proposed by Barabási [29], "independent of their age, function, and scope, converge to similar architectures."

However, we have a dilemma in solving the degree exponent of BDN, which is more close to practical complex networks. Besides, the constructing mechanism is required to be more concrete to simulate the reality. Since the network modeling and degree distribution are the key issues for complex networks, it is worth exploring for the perfect solution to build fit models and seek for their degree exponents.

REFERENCES

[1] D. J. Watts and S. H. Strogatz, "Collective dynamic of 'small-world' networks," *Nature*, vol. 393, pp. 440–442, Jun. 1999.

[2] M. E. J. Newman and D. J. Watts, "Renormalization group analysis of the small-world network model," *Phys. Lett. A*, vol. 263, no. 6, pp. 341–346, 1999.

[3] A. L. Barabási and R. Albert, "Emergence of scaling in random networks," *Science*, vol. 286, pp. 509–512, Oct. 1999.

[4] A. L. Barabási, R. Albert, and H. Jeong, "Mean-field theory for scale-free random networks," *Phys. A*, vol. 272, nos. 1–2, pp. 173–187, 1999.

[5] P. L. Krapivsky, S. Render, and F. Leyvraz, "Connectivity of growing random networks," *Phys. Rev. Lett.*, vol. 85, no. 4629, pp. 1–4, 2000.

[6] S. N. Dorogovtsev, J. F. F. Mendes, and A. N. Samukhin, "Structure of growth networks with preferential linking," *Phys. Rev. Lett.*, vol. 85, no. 4633, pp. 1–9, 2000.

[7] P. Holme and J. Saramäki, "Temporal networks," *Phys. Rep.*, vol. 519, no. 3, pp. 97–125, 2012.

[8] M. Barthélemy, "Spatial networks," *Phys. Rep.*, vol. 499, nos. 1–3, pp. 1–101, 2011.

[9] S. N. Dorogovtsev and J. F. F. Mendes, "Evolution of networks with aging of sites," *Phys. Rev. E*, vol. 62, no. 2, pp. 1842–1845, 2000.

[10] S. N. Dorogovtsev and J. F. F. Mendes, "Effect of the accelerating growth of communication networks on their structure," *Phys. Rev. E*, vol. 63, Feb. 2001, Art. ID 025101.

[11] Y. B. Xie, T. Zhou, and B. H. Wang, "Scale-free networks without growth," *Phys. A*, vol. 387, no. 7, pp. 1683–1688, 2008.

[12] T. Zhou, G. Yan, and B. H. Wang, "Maximal planar networks with large clustering coefficient and power-law degree distribution," *Phys. Rev. E*, vol. 71, Apr. 2005, Art. ID 046141.

[13] G. Wen, W. Yu, M. Z. Q. Chen, X. Yu, and G. Chen, "$\mathcal{H}_\infty$ pinning synchronization of directed networks with aperiodic sampled-data communications," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 61, no. 11, pp. 3245–3255, Nov. 2014.

[14] W. Lian, R. G. Dromey, and D. Kirk, "Software engineering and scale-free networks," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 39, no. 3, pp. 648–657, Aug. 2009.

[15] W. Yu *et al.*, "Local synchronization of a complex network model," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 39, no. 1, pp. 230–241, Feb. 2009.

[16] M. C. Spencer *et al.*, "Multiscale evolving complex network model of functional connectivity in neuronal cultures," *IEEE Trans. Biomed. Eng.*, vol. 59, no. 1, pp. 30–34, Jan. 2012.

[17] H. Su *et al.*, "Decentralized adaptive pinning control for cluster synchronization of complex dynamical networks," *IEEE Trans. Cybern.*, vol. 43, no. 1, pp. 394–399, Feb. 2013.

[18] G. C. Chasparis, "Network formation: Neighborhood structures, establishment costs, and distributed learning," *IEEE Trans. Cybern.*, vol. 43, no. 6, pp. 1950–1962, Dec. 2013.

[19] M. N. Schmidt and M. Morup, "Nonparametric Bayesian modeling of complex networks: An introduction," *IEEE Signal Process. Mag.*, vol. 30, no. 3, pp. 110–128, May 2013.

[20] N. Mahdavi, M. B. Menhaj, J. Kurths, and J. Lu, "Fuzzy complex dynamical networks and its synchronization," *IEEE Trans. Cybern.*, vol. 43, no. 2, pp. 2168–2267, Apr. 2013.

[21] M. E. J. Newman, "Finding community structure in networks using the eigenvectors of matrices," *Phys. Rev. E*, vol. 74, Sep. 2006, Art. ID 036104.

[22] H. Jeong, B. Tombor, R. Albert, Z. N. Oltvai, and A.-L. Barabási, "The large-scale organization of metabolic networks," *Nature*, vol. 407, pp. 651–654, Oct. 2000.

[23] M. L. Goldstein, S. A. Morris, and G. G. Yen, "Problems with fitting to the power-law distribution," *Eur. Phys. J. B-Condens. Matter Complex Syst.*, vol. 41, no. 2, pp. 255–258, 2004.

[24] A. Clauset, C. R. Shalizi, and M. E. J. Newman, "Power-law distributions in empirical data," *SIAM Rev.*, vol. 51, no. 4, pp. 661–703, 2009.

[25] Y. Virkar and A. Claust, "Power-law distributions in binned empirical data," *Ann. Appl. Stat.*, vol. 8, no. 1, pp. 89–119, 2014.

[26] M. Feng, H. Qu, and Y. Zhang, "Highest degree likelihood search algorithm using a state transition matrix for complex networks," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 61, no. 10, pp. 2941–2950, Oct. 2014.

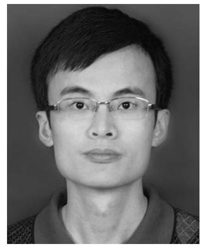
[27] R. Albert, H. Jeong, and A. L. Barabási, "Attack and error tolerance of complex networks," *Nature*, vol. 406, pp. 378–382, Jul. 2000.

[28] J. M. Carlson and J. Doyle, "Highly optimized tolerance: Robustness and design in complex systems," *Phys. Rev. Lett.*, vol. 84, no. 11, pp. 2529–2532, 2000.

[29] A. L. Barabási, "Scale-free networks: A decade and beyond," *Science*, vol. 325, pp. 412–413, Jul. 2009.

[30] P. Erdos and A. Rényi, "On random graphs," *Publ. Math. Debrecen*, vol. 6, pp. 290-Ű297, 1959.

[31] R. K. Merton and K. Robert, "The Matthew effect in science," *Science*, vol. 159, pp. 56Ű-63, Jan. 1968.



Minyu Feng received the B.S. and M.S. degrees with the School of Mathematical Science and Computer Science and Engineering, University of Electronic Science and Technology of China, Chengdu, China, in 2010 and 2013, respectively, where he is currently pursuing the Ph.D. degree.

His current research interests include network topology, random theory, complex networks, optimization, and machine learning.

Hong Qu (M'09) received the Ph.D. degree in computer science from the University of Electronic Science and Technology of China, Chengdu, China, in 2006.

He is a Professor with the School of Computer Science and Engineering, University of Electronic Science and Technology of China. He is currently a Visiting Scholar with the Potsdam Institute for Climate Impact Research, Potsdam, Germany, and Humboldt University of Berlin, Berlin, Germany. His current research interests include neural networks, deep learning, intelligent computation, and optimization.

Zhang Yi (M'08-SM'09) received the Ph.D. degree in mathematics from the Institute of Mathematics, Chinese Academy of Science, Beijing, China, in 1994.

He is currently a Professor with the College of Computer Science, Sichuan University, Chengdu, China. His current research interests include neural networks, big data, and machine learning.

Xiurui Xie is currently pursuing the Ph.D. degree with the Department of computer science and engineering, University of Electronic Science and Technology of China, Chengdu, China.

Her current research interests include neural networks, intelligent computation, and optimization.

Jürgen Kurths received the B.S. degree in mathematics from the University of Rostock, Rostock, Germany, the Ph.D. degree from the Academy of Sciences of the German Democratic Republic, Berlin, Germany, in 1983, and the Honorary degrees from the N. I. Lobachevsky State University of Nizhny Novgorod, Nizhny Novgorod, Russia, and Saratow State University, Saratow, Russia, in 2008 and 2012, respectively.

From 1994 to 2008, he was a Full Professor with the University of Potsdam, Potsdam, Germany. Since 2008, he has been a Professor of Nonlinear Dynamics with Humboldt University, Berlin, and the Chair of the Research Domain Transdisciplinary Concepts with the Potsdam Institute for Climate Impact Research, Potsdam. Since 2009, he has been the Sixth-Century Chair with Aberdeen University, Aberdeen, U.K. His current research interests include synchronization, complex networks, time series analysis, and their applications. He has authored over 480 papers with 18 000+ citations and an H-factor of 56.

Dr. Kurths was a recipient of the Alexander von Humboldt Research Award from the Council of Scientific and International Research, India. He is an Editor of *PLoS ONE*, *Philosophical Transactions of the Royal Society A*, and *Chaos*. He is a fellow of the American Physical Society and a member of the Academia Europea and the Macedonian Academy of Sciences and Arts.