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Minyu Feng, Liangjian Deng and Jürgen Kurths

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Evolving networks based on birth and death process regarding the scale stationarity

Minyu Feng,[1,a)] Liangjian Deng,[2] and Jürgen Kurths[3,b)]
[1]*School of Computer Science and Engineering, University of Electronic Science and Technology of China, Chengdu 610054, China*
[2]*School of Mathematical Sciences,University of Electronic Science and Technology of China, Chengdu 611731, China*
[3]*Potsdam Institute for Climate Impact Research, 14473 Potsdam, Germany*

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Since the past few decades, scale-free networks have played an important role in studying the topologies of systems in the real world. From the traditional perspective, the scale of network, the number of nodes, keeps growing over time without decreasing, leading to the non-stationarity of the scale which is against the real networks. To address this issue, in this paper, we introduce both increase and decrease of vertices to build the evolving network models based on birth and death random processes which are regarded as queuing systems in mathematics. Besides the modeling, the scale of networks based on different random processes is also deduced to be stationary and denoted by a specific probabilistic expression irrelevant to time. In the simulations, we build our network models by different types of queueing systems and compare the statistical results with theories to show the validity and accuracy of our proposed models. Additionally, our model is applied to simulate and predict the populations of some developed countries in recent years. *Published by AIP Publishing.*
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The modeling of the complex networks with both birth and death processes is a challenging task for the network science. The task is to properly build these networks and describe them in a mathematical expression. The death process significantly affects the structure and topologies of the complex networks, but it is often ignored. The presented paper shows a novel network model regarding both the birth and death of vertices in the network, utilizes the different queueing systems to express the models, and calculates their statistical results. Based on this model, we define the number of vertices as the scale of a network that will stay stationary as time goes by, which can be considered as a topological property of complex networks. Our simulations suggest that the scale of this model can describe the population network effectively.

I. INTRODUCTION

At the end of the last century, the proposition of the conception of scale-free (SF) networks successfully created a sensation, and the researchers in multidisciplinary fields become aware of the importance of network science and gradually made significant achievements. As we know, many communication, social, and biological systems regarded as complex networks. For better understanding real networks, utilizing the reasonable models to simulate these real-world networks is a significant issue considered as the primary research for network science. Besides, the topology properties which stay constant in the continuous process significantly influence our cognition for the network structure.

In this paper, we focus on the network modeling. Various network models along with the research of their topologies were springing up after the SF networks were proposed. It is suggested that the variation on connections and vertices is important for network modeling. Developing and decaying networks[2] and accelerating growth networks were the early models to notice the connection variety. Recently, a model based on the age of vertices[3] was proposed, and it studied the influence of age on the network structure. Soon, a network with variable arrival rate[4] was proposed and further proved that the rate is independent of the degree distribution. Besides, the rule of connection is also considered as the key of a network model. The well-known local-world evolving network,[5] and its latest extension model like the neighborhood log-on and log-off model,[6] and weighted local-world evolving network[7] presented the relationship of vertices based on the local priority. There are other network models considering more complex facts. The dynamical network models based on varying time[8] showed the temporal influence; another complex network model was constructed by a time series which inherits the main properties of the time series in its structure,[9] and nonparametric Bayesian networks[10] made it possible to specify flexible model structures and infer the adequate model complexity from the observed data, and a statistical mechanics approach was used for the description of complex networks.[11] Recently, some researchers noticed that the networks keep not just growing but decreasing in the meantime, for example, a theoretical random network studied both the death and birth of vertices[12] and an arbitrary symmetric complex network with delay-coupled oscillators is investigated by the oscillation death.[13] Based on these network models, topology properties that display a stationarity during the growing process of networks are usually discussed. The Power-law degree distribution is considered as the main

[a)]Electronic mail: fmy1987@sina.com
[b)]Also at Department of Physics, Humboldt University, Berlin 12489, Germany.

topology property for the SF networks which is independent of time.[14] Then, other researchers tried numerical methods to give an analytical expression for different networks, and the most recent method used the fastest mixing reversible Markov chains.[15] Besides, the average path length is another topology property frequently investigated. Newman *et al.* first calculate the average path length of both small-world models by the renormalization group method[16] and mean field solution.[17] Other properties such as the coreness,[18,19] betweenness,[20] and influential node[21] have been discussed animatedly in recent years. All their pioneering works on topology properties provided the idea to solve topology properties by the probabilistic method.

To propose a novel evolving network model and study its potential topology property, in this paper, based on the existing study, we introduce the birth process and death process to the evolving network modeling for simulating those real networks driven by growth and decrease. To better understand the process, the queueing systems are creatively adopted to describe all vertices in the network as the customers, their birth process as the input process, and their stays in the network as a service process. Different from the description of "scale" in the SF networks, we define the scale of networks as the number of vertices in network and find that it will stay stationary as time goes by, which is considered as a typical topology property. Inspired by other studies on topology properties, we prove the existence of stationary network scale and offer its analytic expression by the probability theory. In the simulation, we compare the statistical results of our proposed models with theoretical results. In the realistic perspective, we also compare our models with the population of some developed countries during the recent 55 years, indicating that our model is potentially able to predict the population.

The organization of this paper is as follows: A detailed presentation of the network model based on birth and death process and its theoretical study is provided in Sec. II. Simulations are carried out in Sec. III to demonstrate the validity and accuracy of our model to the theoretical result and real-world population. Finally, some discussions, conclusions, and outlooks are given in Sec. IV.

II. COMPLEX NETWORKS BASED ON BIRTH AND DEATH PROCESS

In contrast with traditional growing networks, we introduce a novel modeling method based on birth and death process applying the queueing theory. As we mentioned above, most complex networks have a distinct birth and death phenomenon displaying the arrival of new vertices and the extinction of obsolete vertices.

If there are n vertices in the network, new arrivals enter the network at an exponential rate λ_n, and obsolete vertices leave the system at a specific rate μ_n; we consider that this network follows a birth and death process. Furthermore, suppose that each vertex in the network is a customer and the mechanism of the extinction is the server, then the network itself can be described as a queueing system. Once a vertex has been served, it is dead and wiped from the network. In the perspective of queueing theory, the number of vertices in the network is denoted as the queue length, the waiting time covers from the arrival of a vertex till it gets service, and the lifespan of a vertex is the sum of the waiting time and service time. As the waiting time is long enough, the service time can be ignored. Based on these assumptions, we next present the modeling of networks based on birth and death process (NBD) and its relevant theories.

A. Modeling of NBD

First, we introduce the process of constructing an NBD as an SF network. The generation of vertices follows a Poisson stochastic process with the parameter λ, for each arrival, m links are connected to the existing vertices by the probability that higher degree has the preference. Once being connected, the vertices are regarded as customers under service with infinite servers, and their extinction follows a specific process. As this NBD is a typical queueing system, for clear saying, we introduce this model by defining its basic elements.

Queueing System of NBD

Input process. The events of vertex birth usually occur "at random"; the Poisson process is, therefore, an appropriate model for describing them. For a queueing system of NBD, we assume that the new vertices are input at a rate of λ.

Service mechanism. The network itself can be connected to as many vertices as possible, indicating that the number of vertices under service is without limit, numerically from 0 to ∞. The servers work in parallel which means that all customers are under service at the same time. Besides, the events of vertex death are more complicated than that of birth. Some of them just follow the Poisson process which means that they have exponential service time with the parameter μ, and the population networks is a typical example of a mortality rate. On the other hand, for those NBD in competition, i.e., the economic networks, their service time is not merely exponential, but we generalize that they are distributed independently and identically with a general distribution (i.i.d.) $G(t)$.

System capacity. In a queueing system, the number of customers waiting at a time is generally significant. However, in the practical situation, the servers are always sufficient, waiting time thus is avoided for NBD.

Queueing discipline. The rule that the server accepts customers is mainly considered for the queueing discipline. In this paper, we focus on the rule first come, first serve (FCFS) which is very common for NBD based on the lifespan, such as population networks.

From the queueing system of NBDs, we write their symbolically representing queueing systems as $M/M/\infty$ or $M/G/\infty$ depending on their properties, where M denotes the Poisson arrival or leave, G means the general service, and ∞ is the number of servers.

The queueing system defines the input and output process for vertices of a NBD, but the connection rule is still required to complete the modeling. Hence, we propose the modeling processes of theoretic NBD based on $M/M/\infty$ and $M/G/\infty$

consisting of initial network, growth mechanism, preferential connection, and end time.

Modeling of a NBD

Initialization. Given N vertices form a nearest-neighbor coupled network, and each vertex in this network connects to its $K/2$ neighbors on the left and $K/2$ neighbors on the right. For any pair of vertices, an extra connection is established by the probability of p.

Growth and extinction. As mentioned in the queueing system, the new vertices arrive at a specific rate λ. For each new vertex denoted as i, m edges connect to the existing vertices.

For the $M/M/\infty$ system, the existed vertices are gone by a specific rate μ, denoting that the service time of vertices follows

$$T(t) = 1 - e^{-\mu t}, \quad t \geq 0. \tag{1}$$

For the $M/G/\infty$ system, the service time of existed vertices follows a general distribution (i.i.d.) $G(t)$ with a certain rate

$$\mu = \left[\int_0^\infty t dG(t)\right]^{-1}. \tag{2}$$

Connection and disconnection. The probability of a new arrival connected to an existing vertex is decided by Π_i,

$$\Pi_i = \frac{k_i}{\sum_j k_j}. \tag{3}$$

And once a served vertex is gone, its related connections are released.

Termination. The limitation of time is manually set to T. Once the time is up, the algorithm terminates, and the network is the output.

Generally, the initial network is small but highly connected, e.g., ARPANET, the precursor of the Internet, has only four vertices dispersed in four universities and connected to each other. Based on these considerations, the network is therefore assumed to initialize as a small-world network, especially, the NW small-world model. The growth and extinction is the key process of the model, the input of which follows a Poisson process discussed above. Moreover, for the output, we consider two situations, one of which follows a Poisson process which fits those similar to life-span networks such as the population networks, and the other is universal for those common networks with different death rules, such as survival of the fittest; one of the examples for this kind of network is the collaborate network. For the Connection and Disconnection, to simplify the model, we employ the preferential attachment shown in Eq. (3) to make the high degree individual more likely to be connected. In the case of death of one vertex, we wipe all of its links from the model. Consider that the model is required to control manually, we set a termination time to stop the model construction.

Referred to the queueing system and model process of the NBD, we illustrate the construction process of an NBD as an instant in Fig. 1. The initial network (a) is a small-world network, each vertex connects to its left and right neighbors, and an extra connection is linked. Each new vertex brings two connections to the network. For simplicity, we let the system be $M/M/\infty$, that is, the birth follows a Poisson process with the parameter λ. Meanwhile, the death time follows an exponential distribution with the parameter μ. By constructive time, we can calculate $\lambda = \frac{5}{10}$ for V_6, V_7, V_8, V_9, V_{10}, and $\mu = \frac{4}{10}$ for V_3, V_6, V_2, V_1. The final network (d) displays a nonuniform degree distribution, the poor vertices with the degree below 3 make up a majority part (83.3%) of this network, and the rich ones with the degree upon 3 are in the minority (16.7%), which is commonly referred to as "the Matthew effect" or "the rich-get-richer phenomenon," and also the property of a Power-law distribution for the SF networks.

B. Stationary analysis of NBD scale

As the modeling process of NBD has been illustrated, in this subsection, we propose a new concept of the stationarity of NBD scale, concretely, that is, as time goes by, the probability of network scales tends to be steady under some conditions. These conditions are very significant for studying the stability of NBD which is also the primary analysis of our research.

Before the theoretical analysis is carried out, we first introduce some required notations and definitions. $\{N(t), t \geq 0\}$ denotes a stochastic process of the scale of a NBD, its state space $\Omega = \{1, 2, \ldots\}$, indicating the set of possible values of $N(t)$. We let $p_{mn}(\Delta t)$ denotes the probability that the scale m will next transfer into n in the interval Δt and is expressed as a conditional probability,

$$p_{m,n}(\Delta t) = P\{N(t+\Delta t) = n | N(t) = m\}, \quad m, n \in \Omega. \tag{4}$$

Besides, we let

$$p_n(t) = P\{N(t) = n\}, \quad n \in \Omega \tag{5}$$

denote the probability that the number of network vertices is n at time t. And giving $t \to \infty$, if the probability converges to some values, then define the limiting probability

$$p_n = \lim_{t\to\infty} p_n(t), \quad n \in \Omega \tag{6}$$

as the stationary probability distribution or steady-state probability.

Consider that there are two kinds of potential services as $M/M/\infty$ and $M/G/\infty$, we study each of them in Secs. II B 1 and II B 2.

1. M/M/∞ queueing systems

For the $M/M/\infty$ systems of NBD, we first carry out two lemmas for the proof of stationarity. In the state space Ω, we have

Lemma 1. *Assuming that the limiting probability* p_n *exists,* $n \in \Omega$*, then*

$$\lim_{t\to\infty} p_n(t)' = 0. \tag{7}$$

Proof. Suppose that there exists $x \in \Omega$, that makes $\lim_{t\to\infty} p_x(t)' = s$, in which $s > 0$. By the definition of limitation,

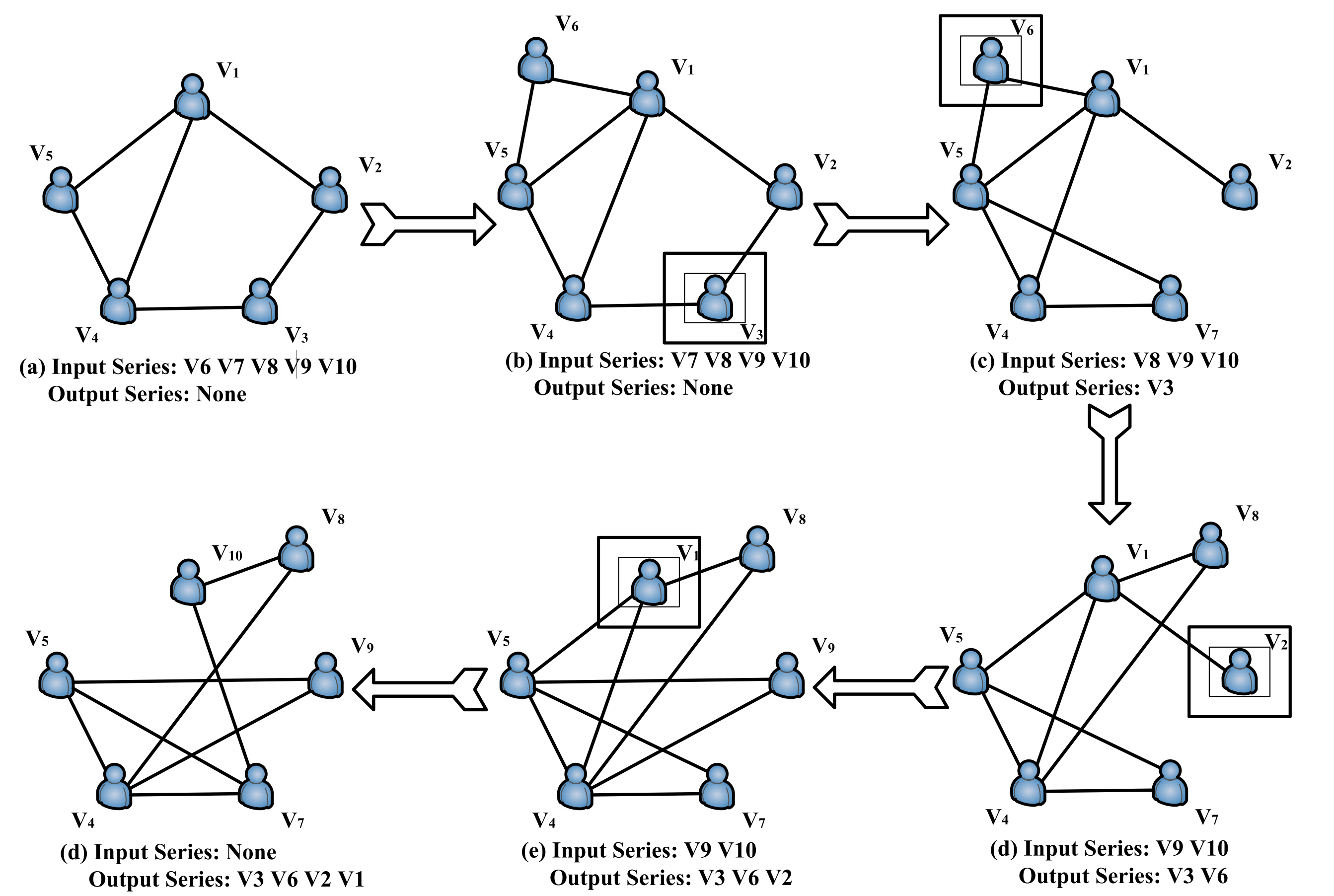

FIG. 1. An illustration of a NBD based on the queueing system of $M/M/\infty$ to show the constructive process for modeling, the constructive time span T of which is supposed from 0 to 10. Each subgraph is intituled the input and output series which denote the birth of new vertices and death of existing vertices, and the vertices in double boxes indicate that they are under service and will output next time. For each new vertex, two connections are linked to the existing vertices.

we know that there exists a t_1, for arbitrary ε and those $t \geq t_1$,

$$| p_x(t)' - s | < \varepsilon \tag{8}$$

indicating that $p_x(t)' > s + \varepsilon$.

Therefore, we can deduce the limitation,

$$\lim_{t\to\infty} p_x(t) = \lim_{t\to\infty}\left[p_x(t_1) + \int_{t_1}^{t} p_x(y)' dy\right]$$
$$> p_x(t_1) + \lim_{t\to\infty}(s+\varepsilon)(t-t_1) = \infty, \tag{9}$$

which is against the definition of probability which is required to be lesser than or equal to 1. Then, $\lim_{t\to\infty} p_x(t)' = 0$.

The result follows. □

Supposing that the arrivals of vertices are assumed to occur in a Poisson process with rate λ and the service times of vertices have an exponential distribution with a parameter μ, then we can say

Lemma 2. *The transition probabilities of NBD based on $M/M/\infty$ systems are*

$$p_{m,n}(\Delta t) = \begin{cases} \lambda\Delta t + o(\Delta t), & m = n+1 \\ m\mu\Delta t + o(\Delta t), & m = n-1 \\ 1 - (\lambda + m\mu)\Delta t + o(\Delta t), & m = n \\ o(\Delta t), & |m-n| \geq 2 \end{cases}. \tag{10}$$

Proof. According to the Poisson process, during Δt, the probability that only one vertex arrives is $\frac{\lambda\Delta t}{1!}e^{-\lambda\Delta t}$, the probability the service is unfinished is $e^{-\mu\Delta t}$. Besides, the probability that n ($n \geq 2$) vertices arrive and $n-1$ vertices have been served is $o(\Delta t)$. Thus, applying Taylor's formula, we yield the transition probability

$$\begin{aligned} P_{m,m+1} &= \frac{\lambda\Delta t}{1!}e^{-\lambda\Delta t}\cdot e^{-\mu\Delta t} + o(\Delta t) \\ &= \lambda\Delta t[1 - (\lambda+\mu)\Delta t + o(\Delta t)] + o(\Delta t) \\ &= \lambda\Delta t + o(\Delta t). \end{aligned} \tag{11}$$

Analogously, during Δt, the probability that none of the new vertices arrives is $e^{-\lambda\Delta t}$, and for m vertices, only one has been served by any one of $i-1$ servers is $\sum_{k-1}^{m}(e^{-\mu\Delta t})^{k-1}(1 - e^{-\mu\Delta t} = 1 - e^{-m\mu\Delta t}$. The probability that $n-1$ ($n \geq 2$) vertices arrive and n have been served is $o(\Delta t)$. Then, we get

$$p_{m,m-1} = e^{-\lambda\Delta t}(1 - e^{-m\mu\Delta t}) + o(\Delta t) = m\mu\Delta t + o(\Delta t). \tag{12}$$

Obviously, the probability that none of the vertices arrives or leaves is

$$p_{m,m} = 1 - (\lambda + m\mu)\Delta t + o(\Delta t) \tag{13}$$

and also for $|m-n| \geq 2$,

$$p_{m,n} = o(\Delta t). \tag{14}$$

Above all, combining Eqs. (11)–(14), the results follow. □

Theorem 1. *For the $M/M/\infty$ systems of NBD, suppose that the initial condition follows that $p_1(0) = 1$ and $p_{n\neq 1}(0) = 0$, let $t \to \infty$, the stationary distributions of scale $\{N(t),\ t \geq 0\}$ exist, and follow*

$$p_n = \frac{\lambda^n}{n!\mu^n} e^{-\frac{\lambda}{\mu}}. \tag{15}$$

Proof. We first prove the existence of stationary distribution, equivalent to proving that $\{N(t),\ t \geq 0\}$ is a homogeneous irreducible continuous Markov chain.

Referring to Eq. (4), we know that the transition probability is only relevant to the time interval Δt but irrelevant to the starting time t, indicating that the chain is homogeneous.

For any $n, m \in \Omega$, according to Lemma 2, there always exists a certain t following $p_{m,n}(t) > 0$; there also exists a t that lets $p_{n,m}(t) > 0$, denoting that all states in Ω are communicated leading to that Ω cannot be divided into a smaller subset. Therefore, the chain is irreducible.

Again to Eq. (4), we can easily derive that $\lim_{t\to o^+} p_{n,n}(t) = 0$ while $\lim_{t\to o^+} p_{m,n\neq m}(t) = 0$, that is

$$\lim_{t\to o^+} p_{m,n}(t) = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}, \tag{16}$$

the condition of continuity holds.

In summary, the chain $\{N(t), t \geq 0\}$ is homogeneous, irreducible, and continuous, so its stationary distribution exists.

As the stationary distribution exists, then we try to solve its explicit expressions.

Consider an extremely small time interval Δt, the variation of probability $p_{mn}(t)$ is expressed as

$$\begin{aligned} \Delta P &= p_{m,n}(t+\Delta t) - p_{m,n}(t) \\ &= \sum_{l\in\Omega} p_{m,l}(t)p_{l,n}(\Delta t) - p_{m,n}(t) \end{aligned} \tag{17}$$

obviously, utilizing the initial condition with the formula of Total Probability, we have

$$\begin{aligned} p_n(t) &= \sum_{m\in\Omega} P\{N(0+\Delta t) = n | N(0) = m\} \\ &= \sum_{m\in\Omega} p_m(0)p_{m,n}(t) = p_{m,n}(t). \end{aligned} \tag{18}$$

Employing this equation, Eq. (17) is rewritten as

$$\begin{aligned} \Delta P &= \sum_{l\in\Omega} p_l(t)p_{l,n}(\Delta t) - p_n(t) = p_{n-1}(t)[\lambda\Delta t + o(\Delta t)] \\ &\quad + p_{n+1}(t)[(n+1)\mu\Delta t + o(\Delta t)] + p_n(t)[1 - (\lambda \\ &\quad + n\mu)\Delta t + o(\Delta t)] + \sum_{|l-n|\geq 2} p_l(t)o(\Delta t) - p_n(t) \\ &= \lambda\Delta t p_{n-1}(t) + (n+1)\mu\Delta t p_{n+1} \\ &\quad - (\lambda + n\mu)\Delta t p_n + o(\Delta t). \end{aligned} \tag{19}$$

Let $\Delta t \to 0$, the formula is denoted as

$$\begin{aligned} p_n'(t) &= \lim_{\Delta t\to 0} \frac{\Delta P}{\Delta t} \\ &= \lambda p_{n-1}(t) + (n+1)\mu p_{n+1}(t) - (\lambda + n\mu)p_n(t), \end{aligned} \tag{20}$$

specifically, for $n = 0$, we have

$$p_0'(t) = -\lambda p_0(t) + \mu p_1(t). \tag{21}$$

For $t \to 0$, by Lemma 1, we obtain the following equations:

$$\begin{cases} \lambda p_0(t) - \mu p_1(t) = 0 \\ \lambda p_{n-1}(t) + (n+1)\mu p_{n+1}(t) - (\lambda + n\mu)p_n(t) = 0 \end{cases}, \tag{22}$$

along with $\sum_{n=0}^{\infty} p_n = 1$, the solution is

$$p_n = \frac{\lambda^n}{n!\mu^n} e^{-\frac{\lambda}{\mu}}. \tag{23}$$

The results follow. □

Additionally, by Eq. (22), we can easily obtain the transient scale distribution of NBD based on $M/M/\infty$ systems

$$p_n(t) = exp\left[-\frac{\lambda}{\mu}(1-e^{-\mu t})\right]\frac{1}{n!}\left[\frac{\lambda}{\mu}(1-e^{-\mu t})\right]^n, \tag{24}$$

which is useful for analyzing other quantities of interest for NBD model.

Theorem 1 gives the steady-state probabilities for the vertex number of networks, by which we can obtain some statistical properties of networks.

Theorem 2. *The average scale of NBD based on the $M/M/\infty$ systems is*

$$E(N) = \frac{\lambda}{\mu} \tag{25}$$

the average staying time of each vertex is

$$E(T) = \frac{1}{\mu} \tag{26}$$

and the standard deviation of the scale is

$$\sigma_N = \sqrt{\frac{\lambda}{\mu}}. \tag{27}$$

Proof. The average scale is also the expectation of the result of Theorem 1 expressed as

$$E(N) = \sum_{n=0}^{\infty} np_n = \frac{\lambda}{\mu} e^{-\frac{\lambda}{\mu}} \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{\mu^{n-1}(n-1)!} = \frac{\lambda}{\mu} \tag{28}$$

according to the Little equation, the staying time is

$$E(T) = \frac{N_{ave}}{\lambda} = \frac{1}{\mu} \tag{29}$$

and the standard deviation is the square root of its variance, thus

$$\begin{aligned} \sigma_N &= \sqrt{E(N^2) - E(N)^2} \\ &= \sqrt{\frac{\lambda}{\mu} e^{-\frac{\lambda}{\mu}} \sum_{n=0}^{\infty} \frac{(n+1)\lambda^n}{\mu^n n!} - \frac{\lambda^2}{\mu^2}} = \sqrt{\frac{\lambda}{\mu}}. \end{aligned} \tag{30}$$

The results follow. □

From Theorem 2, we can estimate the expectation scale of the NBD and also the lifespan of a vertex; besides, the standard deviation is a measure to quantify the amount of variation or dispersion of a set of scales.

2. M/G/∞ queueing systems

Apparently, $M/G/\infty$ queueing systems differ from the death process of $M/M/\infty$, which displays a more general way for the extinguishment of network vertices but makes this process non-Markov; in other words, the methods above based on the Markov chain have no effect. Therefore, we construct a cumulative conditional probability to address the issue.

Suppose that S denotes the service time, and $N(t)$ is the number of vertices at time t, we introduce the process $C(t)$ to accumulate the number of vertices before time t. Then, we have

Lemma 3. *The number of vertices at time t and those that arrive before time t follows the conditional probability*

$$P_{m,n}(N|C,t) = P\{N(t) = n|C(t) = m\} = \begin{cases} \dfrac{m!F(t)^n[t - F(t)]^{m-n}}{t^m n!(m-n)!}, & n \le m \\ 0, & n > m \end{cases}, \quad (31)$$

where $F(t) = \int_0^t [1 - G(t-x)]dx$ *is the complementary distribution and* G *is the probability distribution of* S.

Proof. Consider that the service time is S, at time t, one vertex is still in the network, which means its arrival time x follows that $t - S < x < t$. As we know, the arrival of vertices follows a Poisson process, then the number of vertices that arrive t before time is uniformly distributed in $[0, t]$. Thus, the probability that a vertex which arrives before time t but is still in the network is

$$X(t) = \int_0^t \frac{P[S > t - x]}{t} dx = \int_0^t \frac{1 - G(t-x)}{t} dx. \quad (32)$$

At time t, consider that the arrival time of each vertex is i.i.d., $P_{m,n}(N|C,t)$, the probability of the number of vertices at time t and those that arrive before time t is binomially distributed. Obviously, for those $n > m$,

$$P_{m,n}(N \mid C, t) = 0, \quad (33)$$

otherwise $n \le m$, with $F(t) = tX(t)$,

$$P_{m,n}(N \mid C, t) = \binom{n}{m} X(t)^m [1 - X(t)]^{m-n} = \frac{m!F(t)^n[t - F(t)]^{m-n}}{t^m n!(m-n)!}. \quad (34)$$

The result follows. □

Then, we study the steady probability distributions for those NBD based on $M/G/\infty$ systems.

Theorem 3. *For the* $M/G/\infty$ *systems of NBD, suppose that the expectation of death process* $\{G(t), t \ge 0\}$ *exists, let* $t \to \infty$*, the stationary distributions of scale* $\{N(t), t \ge 0\}$ *exist, and follow*

$$p_n = \frac{\{\lambda \int_0^\infty [1 - G(x)]dx\}^n}{n!} e^{-\lambda \int_0^\infty [1-G(x)]dx}. \quad (35)$$

Proof. First, we prove the existence of stationary distributions. Referring to Lemma 3, $\{A(t), t \ge 0\}$ follows a Poisson process, we employ the total probability formula and get

$$p_n(t) = P_{m,n}(N|C,t)P[C(t) = m] = \sum_{m=n}^{\infty} \frac{m!F(t)^n[t - F(t)]^{m-n}}{t^m n!(m-n)!} \frac{(\lambda t)^n}{n!} = \frac{[\lambda F(t)]^n}{n!} e^{-\lambda F(t)}. \quad (36)$$

Apparently, it is a nonhomogeneous Poisson process. Therefore, its limitation exists can be interpreted as the limitation of $F(t)$ exists,

$$\lim_{t\to\infty} F(t) = \lim_{t\to\infty} \int_0^t [1 - G(t-x)]dx = \int_0^\infty [1 - G(y)]dy = E[G(t)]. \quad (37)$$

And we know that the expectation of $\{G(t), t \ge 0\}$ exists; therefore, the limitation of $p_n(t)$ also exists.

Furthermore, with Eqs. (36) and (37), we have the stationary distributions

$$p_n = \lim_{t\to\infty} p_n(t) = \frac{\{\lambda \int_0^\infty [1 - G(x)]dx\}^n}{n!} e^{-\lambda \int_0^\infty [1-G(x)]dx}. \quad (38)$$

The results follow. □

From the results, we can see that if we let $\mu^{-1} = \int_0^\infty [1 - G(x)]dx$, the results are the same as that of Theorem 1 in form. Also, we deduce some of the statistical properties.

Theorem 4. *The average scale of NBD based on the* $M/G/\infty$ *systems is*

$$E(N) = \lambda \int_0^\infty [1 - G(x)]dx, \quad (39)$$

the average staying time of each vertex is

$$E(T) = \int_0^\infty [1 - G(x)]dx, \quad (40)$$

and the standard deviation of the scale is

$$\sigma_N = \sqrt{\lambda \int_0^\infty [1 - G(x)]dx}. \quad (41)$$

Proof. Similar to Theorem 2, the expectation is expressed as

$$E(N) = \sum_{n=0}^{\infty} n p_n = \lambda \int_0^\infty [1 - G(x)]dx \quad (42)$$

the staying time is the expectation of $\{G(t), t \ge 0\}$ which is deduced in Eq. (37). And the standard deviation of the scale is square root of the expectation

$$\sigma_N = \sqrt{E(N)} = \sqrt{\lambda \int_0^\infty [1 - G(x)]dx}. \quad (43)$$

Therefore, the results follow. □

Apparently, these statistical properties are more general expressions, and $M/M/\infty$ is only one typical type that fits these properties.

From the above derivations, we can conclude that whether NBD of $M/M/\infty$ or its general $M/G/\infty$ systems, the scale will stationarily distribute as a Poisson distribution as the time goes by.

III. SIMULATIONS AND ANALYSES

From this section, we start to verify the statistical properties of NBD by the simulations of proposed models and practical models which are supposed to follow the stationary theory of Sec. II.

First of all, we display an illustration of an NBD model as an example. Considering the model in Sec. II A, we set the initial networks as that $N = 100$, $K = 2$, and $p = 0.5$. For the growth and extinction, we choose the $M/M/\infty$ system, and let $\lambda = 2$, $\mu^{-1} = 100$. For a clear illustration, we set $t = 1500$, and the connection each time $m = 5$. As a result, the network is drawn and shown in Fig. 2. As we can see from this figure, the vertices in the center have a higher degree, while those in the surrounding have a lower degree, and obviously, vertices in lower degree are in the majority, which means that this model is a typical SF network. Besides, the size of vertices denotes their ages in the network. The bigger ones mean a longer stay, which are more easily isolated since their friends cannot possibly stay that long as they do.

A. Simulations of proposed models

In this subsection, we try to simulate different NBD and obtain the statistical results, including the scale, stationary distribution, etc. From the proposed models, they are compared with the theoretical results to verify the validity and accuracy. Furthermore, we use the scale of our models to simulate the number of population of different countries from the practical perspective.

1. M/M/∞ systems

These simulations focus on the study of scales of NBD based on $M/M/\infty$ systems; according to our theoretical analysis, as the time t goes by, they are decided by the input rate λ as well as the output rate μ. Therefore, to simulate these systems, we control three parameters λ, μ, and t, where λ is the parameter of an exponential distribution to generate the temporal series of a Poisson process and the same to μ, t is set as a large number to obtain stationary results.

In the following experiments, we select three couples of (λ, μ), and they are $(2, 50)$, $(2, 100)$, and $(4, 100)$ different in the last and first numbers for contrast, to construct NBD based on $M/M/\infty$ systems. To produce the sequences for birth and death of vertices, we mainly employ the function *poissrnd*().

The results of scale varying with time are shown in Fig. 3(a). For better stationary results by acquiring large enough time, we let t be 10^4 and record the scale of each NBD as time passes by in a half-log coordinate. The reason we utilize the half-log coordinate is to better illustrate the ascent stage and the stationary process of plots.

We can clearly see that the results show that the stationarity of scale exists as Theorem 1 describes, all three datasets finally approach to stationarity after a short time ($t = 10^2$), and they all steadily float within bounds. In addition, as Theorem 2 describes, the expectation value of the number is decided by the product of λ and μ^{-1} which approaches to a stationary value, and the standard deviation is the square root of expectation evaluating the dispersion degree to the expectation, the results of these three datasets precisely follow the rule. In detail, after $t = 10^2$, with the same $\mu^{-1} = 100$, the $\lambda = 4$ marked by purple squares approaches to a twice higher

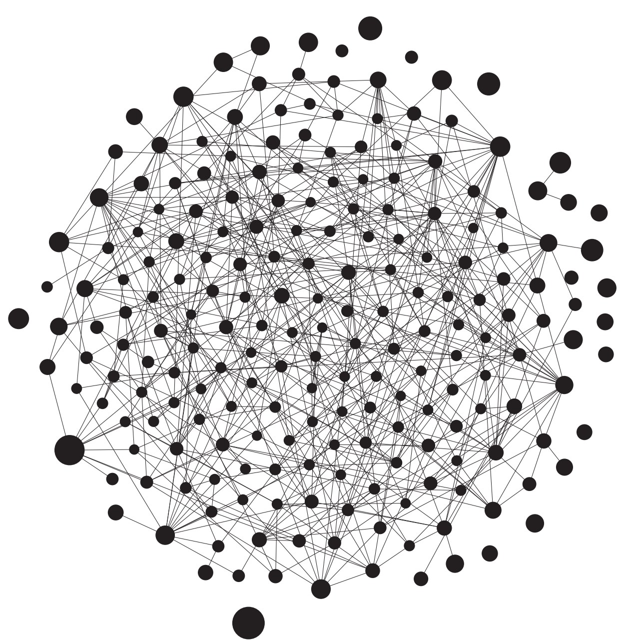

FIG. 2. An illustration of an NBD model based on the queueing system of $M/M/\infty$ at the time of 1500, with the parameters $\lambda = 2$ and $\mu^{-1} = 100$. Each vertex denotes a living individual while the connections reveal their relationships at current time, and its size shows age in the network.

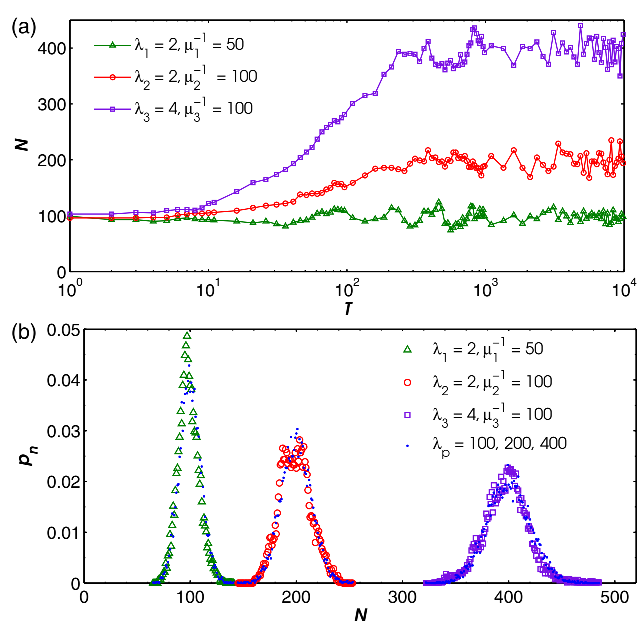

FIG. 3. Statistical results of three datasets with different parameters of NBD models based on $M/M/\infty$. (a) Scale varying with time. (b) Scale statistical distribution and comparison.

TABLE I. The statistical results of scale distributions of NBD with different λ and μ.

	$\lambda\mu^{-1}$		
Properties	2×50	2×100	4×100
Expectation	98.46	198.94	397.54
Standard deviation	9.74	14.57	20.49
Correlation coefficient	97.05%	97.24%	97.75%

value and fluctuates more extensive than $\lambda = 2$ marked by red circles. And with the same $\lambda = 2$, the $\mu^{-1} = 100$ marked by red circles approaches to a twice higher value and fluctuates more extensively than $\mu^{-1} = 50$ marked by green triangles. In other words, the higher λ and μ^{-1} give the result to higher stationary value and dispersion degree.

In addition, the probability distribution of scale is shown in Fig. 3(b). Apart from the figure of scale by time, in this figure, we record the frequency of each scale during the time of 10^4, and by the law of large numbers calculate their percentages as probabilities.

According to our theory, the scale should be a homogeneous Poisson distribution with parameters $\lambda\mu^{-1}$. Obviously, from the results, all three datasets are symmetrical Poisson distributions following the theorem. Furthermore, we can see the expectation value more clearly from this figure. They are 100 marked by green triangles, 200 marked by red circles, and 300 marked by purple squares of central values, which are all products of λ and μ^{-1}. The same to the deviations, with the increase of $\lambda\mu^{-1}$ from left to right, the distributions become wider, which indicates that the deviations are larger. Finally, to further verify our theory, we compare the scale distributions of our models with totally theoretical Poisson distributions. The parameters λ_p of theoretical Poisson distributions are separately setting as 100, 200, and 400 corresponding to the dataset of our models, and shown by blue dots in Fig. 3(b).

The correlation coefficient is utilized to calculate the similarity of our data and theoretical Poisson distribution, which is denoted as

$$\rho_{X,Y} = \frac{E\{[X - E(X)][Y - E(Y)]\}}{\sigma_X \sigma_Y}, \tag{44}$$

where X is the dataset of our models, and Y is the dataset of Poisson distribution. Then, we obtain the correlation coefficient of three comparisons; they are 97.05%, 97.24%, and 97.75%, high enough to show that our scale distributions are very close to the theoretical values, indicating that the scale of our NBD model technically follows a Poisson distribution as our theorem describes.

More concretely, all statistical results are shown in Table I for a clear illustration. Overall, our simulations tell that the scale of NBD model based on $M/M/\infty$ will become stationary as time goes by, and the distribution follows a homogeneous Poisson distribution, which is perfectly in accord with Theorems 1 and 2.

2. M/G/∞ systems

As we described above, there exist many other expressions for death process of individuals except lifespan. Therefore, the $M/G/\infty$ system is applied to denote these more general NBD models. And we try to simulate these NBD models based on the $M/G/\infty$ system in this subsection to show their validity.

In particular, we mainly employ three kinds of distributions for the general distribution G to show different death mechanisms. The first one follows a uniform distribution which denotes that the vertices of NBD are equal to die without distinction (we use R in short to express this simulation), and the parameters a, b are employed to determine the range of uniform distribution. The other one is a log-normal distribution that mainly describes the uneven death mechanism in economics that the rich live while the poor die (L, in short, is to denote this simulation), and the parameters μ, σ are used to describe this distribution. The last simulation focuses on Pareto distribution, also known as Power-law distribution, and is used in the description of social, scientific, geophysical, actuarial, and many other types of observable death mechanisms, which makes the high degree vertices live and low degree ones die (we employ P to denote this simulation), and the parameters α, β are utilized to generate this distribution.

In the following simulations, we still use function *poissrnd()* to generate vertices. For R, we use function *rand()* for death; three couples of (λ, α, β) are selected as simulated objection, and they are (2, 40, 60), (2, 80, 100), and (4, 80, 100). For L, function *lognrnd()* is employed to determine death, we also simulate three couples of (λ, μ, σ), which are, respectively, (2, 3.42, 1), (2, 4.11, 1), and (4, 4.11, 1). And for P, we apply function *pareto()* to simulate three couples of (λ, x_{min}, k), which are (2, 25, 3), (2, 50, 3), and (4, 50, 3) in detail.

The results of R, L, P are displayed in Figs. 4(a), 5(a), and 6(a). Once again, for all simulations of R, L, P, we let t be 10^4 to guarantee that the time is large enough and plot the scales of all NBD varying with time in a half-log coordinate for a clear illustration.

According to Theorem 3, as the time goes by, the stationary scale distribution of NBD with a general death process exists. Obviously, from the results, we can say that all three sets of three different distributions including uniform, log-normal, and Pareto, regarded as general distributions for death process, finally stay stationary after a short time (approximately $t = 10^{-2}$). Considering Theorem 4, the expectation of three different distributions has different results based on their probability distribution functions $G(t)$, and furthermore, their parameters. As a result, the expectations particular for each death distribution in turn are $E(N_R) = \frac{1}{2}\lambda(a + b)$ for R, $E(N_L) = \lambda e^{\mu + \frac{\sigma^2}{2}}$ for L, and $E(N_P) = \frac{\lambda\alpha\beta}{\beta - 1.}$ for P. Similarly, the standard deviations are the square root of their expectations according to Theorem 1. For each distribution, we also employ three different datasets with the same λ but different parameters, and the same parameters but different λ just like the above simulation. The results perfectly follow our theorems, and the datasets of all kinds of distributions

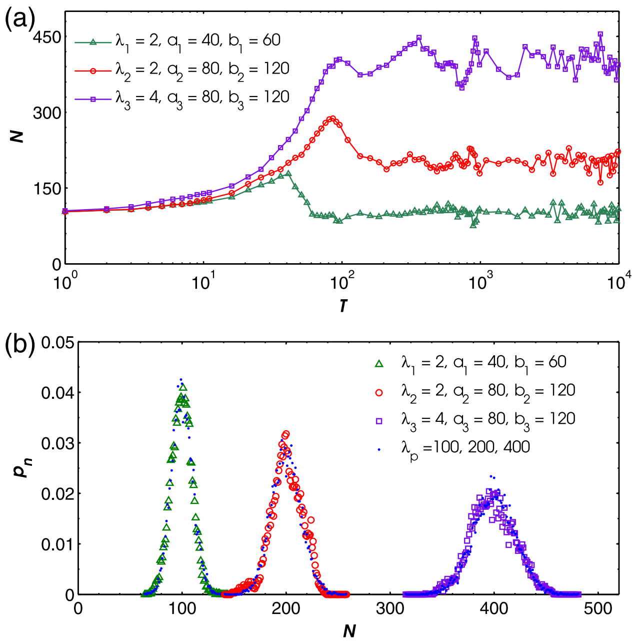

FIG. 4. Statistical results of three datasets with different parameters of NBD models based on specific $M/G/\infty$ employing uniform distribution for service. (a) Scale varying with time. (b) Scale statistical distribution and comparison.

in purple squares have a twice larger stationary value and fluctuate more extensively than those in red circles, while the dataset in red circles have a twice larger stationary value and fluctuate more extensively than those in green triangles. That is to say, the stationary value and dispersion degree are indeed decided by λ and corresponding distribution parameters.

Moreover, diagrams of scale distribution are shown in Figs. 4(b), 5(b), and 6(b). The frequencies of all possible

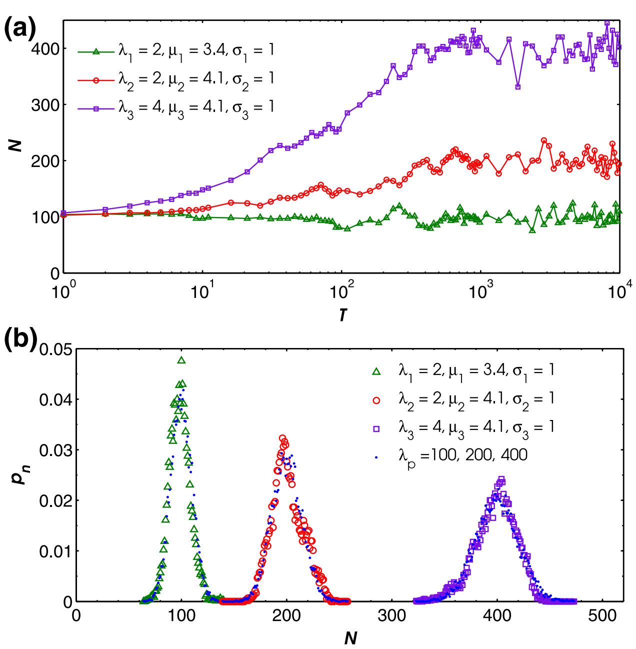

FIG. 5. Statistical results of three datasets with different parameters of NBD models based on specific $M/G/\infty$ employing log-normal distribution for service. (a) Scale varying with time. (b) Scale statistical distribution and comparison.

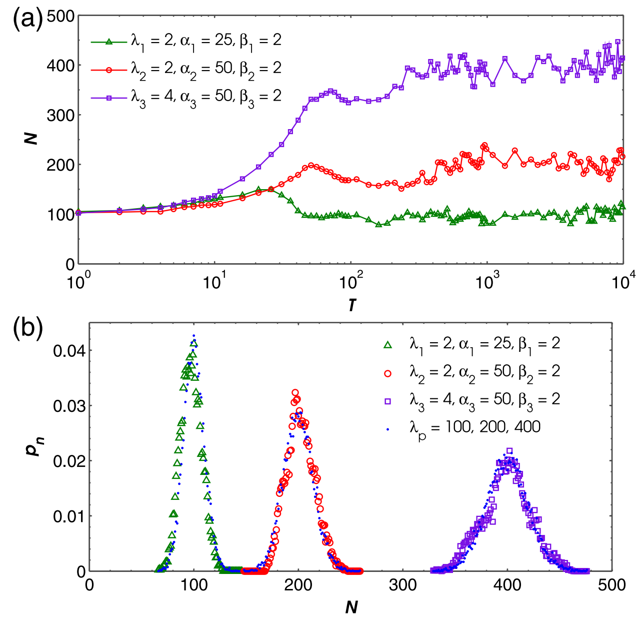

FIG. 6. Statistical results of three datasets with different parameters of NBD models based on specific $M/G/\infty$ employing Pareto distribution for service. (a) Scale varying with time. (b) Scale statistical distribution and comparison.

scales for three $M/G/\infty$ models during the time of 10^4 are regarded as their scale probabilities based on the law of large number.

As we prove above in Theorem 3, the scale of the NBD model based on $M/G/\infty$ should follow a special Poisson distribution with parameters $\lambda\int_0^\infty[1-G(x)]dx$, which means that the parameters are different from three datasets. Specifically, for R, the parameter is $\frac{1}{2}\lambda(a+b)$, for L, it is $\lambda e^{\mu+\frac{\sigma^2}{2}}$, and for P, it is $\frac{\lambda\alpha\beta}{\beta-1}$. We can see that the parameters are also the expectations which is a property of Poisson distribution. Apparently, all figures display that the dataset follows Poisson distributions. For R, the medium values also as expectations are 100 for green triangles, 200 for red circles, and 300 for purple square, which are products of λ and $\frac{1}{2}(a+b)$. For L and P, the results are the same, but the product of λ and $e^{\mu+\frac{\sigma^2}{2}}$, as well as λ and $\frac{\alpha\beta}{\beta-1}$. Besides, with the rise of values of parameters, the curves become wider, indicating that the variances get larger which again follows Theorem 4. We also compare the scale distributions of three datasets with totally theoretical Poisson distribution to demonstrate the correctness of our theory. The parameters λ_p of theoretical Poisson distributions are 100, 200, and 400 corresponding to the datasets of all models and are shown by blue dots in Figs. 4(b), 5(b), and 6(b).

TABLE II. The statistical results of scale distributions of NBD with different λ and $\frac{1}{2}(a+b)$.

	$\lambda\frac{1}{2}(a+b)$		
Properties	2×50	$2^*\times 100$	4×100
Expectation	99.36	200.64	398.71
Standard deviation	9.92	14.81	20.80
Correlation coefficient	99.29%	97.51%	97.42%

TABLE III. The statistical results of scale distributions of NBD with different λ and $e^{\mu+\frac{\sigma^2}{2}}$.

	$\lambda e^{\mu+\frac{\sigma^2}{2}}$		
Properties	2×50	2×100	4×100
Expectation	98.50	200.98	398.48
Standard deviation	9.66	14.76	19.25
Correlation coefficient	97.73%	96.77%	98.09%

We then calculate the correlation coefficient of our datasets with the theoretical Poisson distributions. For R, the results are 99.29%, 97.51%, and 97.42%, for L, they are 97.73%, 96.77%, and 98.09%, while for P, they are 95.88%, 98.50%, and 97.44%. All are very close to the theoretical values, that is to say, all three models based on the $M/G/\infty$ system follow the Poisson distribution.

All the results are listed in Tables II–IV. From all results, we can apparently see that the scale of NBD based on the $M/G/\infty$ system stays stationary by time and finally follows a homogeneous Poisson distribution. It is in line with the $M/M/\infty$ system, which is only a specific kind of the $M/G/\infty$ system. Technically, the scale distribution kind is independent of the distribution type of death, as long as its expectation exists.

B. Simulation of population network

In this subsection, from the perspective of real worlds, we utilize the scale of our NBD model to simulate the populations of three different countries. As we know, the population network can be regarded as NBD, since it has a stationary birth rate as well as an expected average age for every individual. Thus, once we have the datasets of birth rate and average age, we can employ NBD based on $M/M/\infty$ to simulate the population. To get rid of other influencing factors like wars and diseases, we select three developed countries, the Slovak Republic, Poland, and Japan, from the database of the world bank.

In the database of the world bank, we can obtain the population from 1960 to 2015, total 55 years of these three countries[22] to be simulated, which is displayed in Fig. 7 (red circles). Also, the crude birth rates per 1000 people[23] along with life expectancy[24] for these 55 years are referred to as the parameters for our simulation. As we know, the formula for the crude birth rates per 1000 people is denoted as

$$CBR = \frac{\lambda}{p}1000, \tag{45}$$

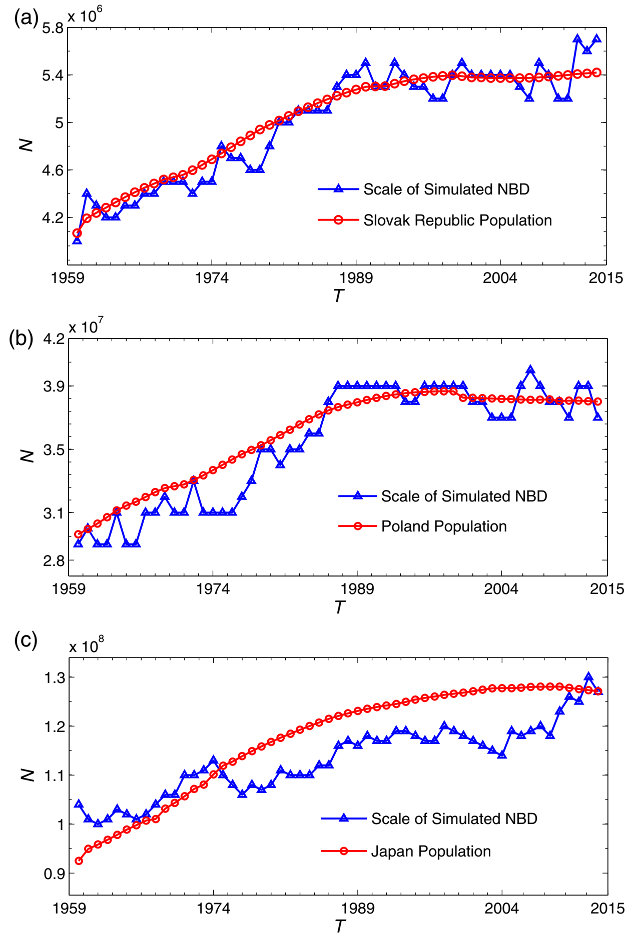

FIG. 7. NBD models based on $M/M/\infty$ system to simulate the populations of three developed countries the Slovak Republic, Poland, and Japan. (a) Simulation for the Slovak Republic. (b) Simulation for Poland. (c) Simulation for Japan.

where p is the population for the specific year. Then, we can obtain the birth rate λ for each year and calculate the average rate for the whole 55 years. Similarly, we can get the average age as the called staying time by calculating the life expectancies for 55 years. The results of parameters for the Slovak Republic, Poland, and Japan are shown in Table V.

After the parameters are obtained, we consider the fact that during the 55 years, these three countries are stationarily developed and regard the result of birth and death rate as the population according to our theory. Then, we employ the proposed NBD based on the $M/M/\infty$ system with the parameters in Table V. These NBD are simulated for 500 years, and in which, 55 years from ascending to stationarity are selected to simulate the years from 1960 to 2015. The

TABLE IV. The statistical results of scale distributions of NBD with different λ and $\frac{\alpha\beta}{\beta-1}$.

	$\lambda\frac{\alpha\beta}{\beta-1}$		
Properties	2×50	2×100	4×100
Expectation	97.37	202.10	399.28
Standard deviation	9.99	13.57	22.00
Correlation coefficient	95.88%	98.50%	97.44%

TABLE V. The parameters of average birth rate λ and average age μ^{-1} for three developed countries.

	Countries		
Parameters	Slovak	Poland	Japan
λ	7.56×10^4	5.25×10^5	1.44×10^6
μ^{-1}	71.93	71.87	77.34

TABLE VI. The comparison between theoretical expectation and real average stationary population.

	Results		
Countries	Expectation	Population[a]	Error[b]
Slovak	5.44×10^6	5.38×10^6	1.12%
Poland	3.77×10^7	3.83×10^7	1.57%
Japan	1.11×10^8	1.27×10^8	12.60%

[a]The population here denotes the average population from 1995 to 2015 considered as the stationary value.

[a]Error here means the relative error.

results are displayed and compared with a real dataset in Fig. 7 (blue triangles).

Equation (44) is employed to get the similarity of our simulated dataset and real dataset. The results are, respectively, 96, 17%, 94.48%, and 88.44% for the Slovak Republic, Poland, and Japan, which indicate that our proposed models are well served to simulate the population. With the rise of λ and μ^{-1} of three countries, the variance also grows, according to Theorem 2, resulting in higher instability and lower fitness.

Besides, we can also use the parameters to predict the expectation of the stationary population in the application of Theorem 2. To show the accuracy of prediction, we first collect the populations from 1995 to 2015 (the populations of three countries stay stationary during this time), 21 years in total and calculate their average as the stationary value. Then, by applying Theorem 2, we can obtain the expectation of the populations as the prediction. Finally, the relative error is used to confirm the accuracy of prediction.

The results are shown in Table VI, and the errors of three countries are, respectively, 1.12%, 1.57%, and 12.60%. Obviously, the lower value of λ and μ^{-1} indicates a lower variance and better prediction, like the Slovak Republic and Poland, and they are all precise with the error near 1%. But with a larger population like Japan, the variance increases radically, which makes the prediction less accurate. Above all, our theoretical expectation can predict the real stationary population within a certain range of error.

From this simulation, we can conclude that our proposed model can also be applied to the population of those developed countries already becoming stationary, and their populations are regarded as the scales of our models essentially. And the expectation can be used to predict the stationary population, especially for those small countries with a low population.

IV. CONCLUSION AND OUTLOOK

The network modeling and its topology property are a highlight research target for network science. In this paper, we introduce the birth and death process to the network modeling and achieve the evolving network described by queueing systems, which is more suited for simulating practical networks with increasing and decreasing individuals than the previous models. More importantly, we primarily define the scale to be a topology property, as it will be stationary as time goes by. Utilizing the probability method inspired by the existing studies, not only do we prove the stationarity of the NBD scale, but its distribution is also confirmed as the Poisson distribution. Besides, we discuss the different types of death mechanisms to fit various networks and conclude that they are independent of the final distribution type of scale. From the simulation results, we learn that the NBD model is in line with the theoretical conclusion and can also be extended to simulate and predict the real population varying with time.

However, there are other situations for networks and may only be considered as the $M/M/\infty$ system or $M/G/\infty$ system. For example, when the capacity of a network is limited, the system capacity cannot be considered as infinity, the birth rate may also be irregular indicating the Poisson process is dysfunctional, etc. For these situations, the scale distribution is more likely different from Poisson distribution. Besides, other factors can also be considered in the modeling of NBD and not only the birth and death. And from the practical perspective, how to apply our model to more precisely simulated population and other stationary network is still worth exploring. All these issues require further work and will be our research goal for the next stage.

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[23]See http://data.worldbank.org/indicator/SP.DYN.CBRT.IN?locations=SK, http://data.worldbank.org/indicator/SP.DYN.CBRT.IN?locations=PL, and http://data.worldbank.org/indicator/SP.DYN.CBRT.IN?locations=JP, respectively, for crude birth rate for the Slovak Republic, Poland, and Japan.
[24]See http://data.worldbank.org/indicator/SP.DYN.LE00.IN?locations=SK, http://data.worldbank.org/indicator/SP.DYN.LE00.IN?locations=PL, and http://data.worldbank.org/indicator/SP.DYN.LE00.IN?locations=JP for life expectancy for the Slovak Republic, Poland, and Japan.